On Some Mixture Models and their Extreme Value Behavior

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1. Introduction

- Finite Mixture: Distribution functions of failure times in lifetime data analysis and reliability engineering.

$$f(x|\Theta) = \sum_{i=1}^{n} c_i f_i(x|\theta_i)$$

- Linear Mixture: Completion times for jobs consisting of the parallel-processing of tasks

$$F^{(k)}(x) = \sum_{i=1}^{k} c_i F_i(x)$$

Layer Model

- Layer Mixture: Modeling individual claim amounts covered by the ceding and reinsurance companies. For example, in an excess of loss contract with a limiting level (threshold), L and a retention level, M, $M \leq L$, the amount,

$$W_i^c = \min(W_i, M) + \max(0, W_i - L)$$

covered by the cedent from each individual claim W_i, $i = 1, 2, \ldots$, with generic distribution $F_W(x)$, has a distribution function

$$F_{W^c}(x) = I_{\{x<M\}} F_W(x) + I_{\{M \leq x\}} F_W(L - M + x)$$

Layer Model

- More general (re)insurance applications: Individual claim amounts X have a mixture distribution, i.e.

$$\begin{array}{llll} X \sim F_1 & \text{for} & X \geq u, & \text{claims below a threshold} \\ X \sim F_2 & \text{for} & X < u, & \text{claims above a threshold} \end{array}$$

(C. Behrens, H. Lopes, and D. Gamerman, 2004)

2. Three Mixture Distribution Models

Layer Mixture Distribution

Definition Given distributions $\{F_i\}_{i=1,2,\ldots}$ and thresholds $0 = u_0 < u_1 < \ldots$, the layer mixture of the first k distributions denoted by $F^{(k)}$ is defined recursively as

$$\overline{F^{(k)}}(x) = I_{\{x<u_{k-1}\}}(x)\overline{F^{(k-1)}}(x) + I_{\{x\geq u_{k-1}\}}(x)\overline{F^{(k-1)}}(u_{k-1})\overline{F}_k(x - u_{k-1})$$

for any integer $k > 1$ and $F^{(1)}(x) = F_1(x)$.

Layer Mixture Distribution

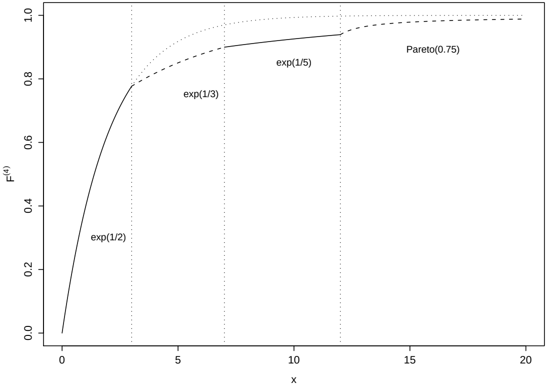

Figure: Layer Mixture

Linear Mixture Distribution

Definition Given distributions $\{F_i\}_{i=1,2,\dots}$, the linear mixture of k distributions denoted by $F^{(k)}$ is defined by

$$F^{(k)}(x) = \sum_{i=1}^{k} c_i F_i(x)$$

where c_i are negative or positive constants that sum to 1.

Conditional Layer Mixture Distribution

Definition Given distributions $\{F_i\}_{i=1,2,\ldots}$ and thresholds $0 = u_0 < u_1 < \ldots$, the conditional layer mixture of the first k distributions denoted by $F^{(k)}$ is defined recursively as

$$\overline{F^{(k)}}(x) = I_{\{x<u_{k-1}\}}(x)\overline{F^{(k-1)}}(x) + I_{\{x\geq u_{k-1}\}}(x)\overline{F^{(k-1)}}(u_{k-1})\overline{F}_k(x|x \geq u_{k-1})$$

for any integer $k > 1$ and $F^{(1)}(x) = F_1(x)$.

Conditional Layer Mixture Distribution

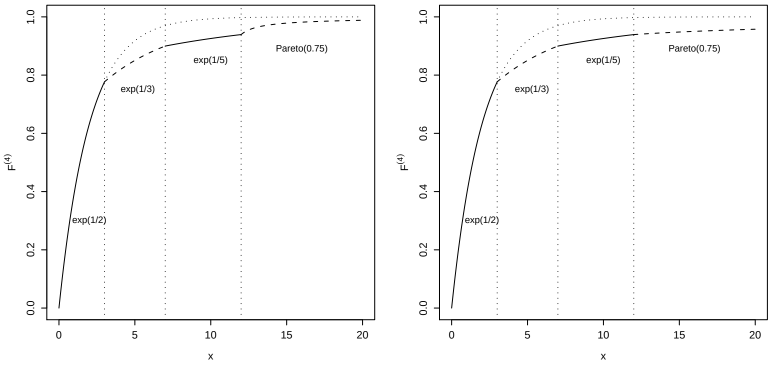

Figure: Layer Mixture vs. Conditional Layer Mixture

3. Maximum Domain of Attraction of Mixture Distributions

MDA of Layer Mixture Distributions

Let $F^{(k)}$ be a layer mixture distribution with distributional components F_i and thresholds $0 = u_0 < u_1 < u_2 < \cdots$ for a positive integer $k > 1$. If $F_k \in MDA(H)$ for an extreme value distribution H, then $F^{(k)} \in MDA(H)$ again and the normalizing constants are

$$\begin{cases} a_n = \gamma^{1/\alpha} a_n^*, \quad b_n = u_{k-1}, & \text{if } H \text{ is Fréchet}, \\ a_n = a_n^*, \quad b_n = b_n^* + u_{k-1} + a_n^* \log \gamma, & \text{if } H \text{ is Gumbel}, \\ a_n = \gamma^{-1/\alpha} a_n^*, \quad b_n = u_{k-1}, & \text{if } H \text{ is Weibull}. \end{cases}$$

where a_n^* and b_n^* are the normalizing constants of $F_k(x)$, $x_n^* = a_n^* x + b_n^*$, and

$$\gamma = \prod_{i=1}^{k-1} r_i \quad \text{where} \quad r_i = \overline{F}_i(u_i - u_{i-1}).$$

MDA of Linear Mixture Distributions

Theorem(Kang and Serfozo, 1999) Suppose that $F^{(k)}$ is a linear mixture distribution and there exists a distribution F^* satisfying

$$\lim_{x \to \infty} \frac{\overline{F_i}(x)}{\overline{F^*}(x)} = r_i$$

for each i. Let $\gamma = \sum_{i=1}^{k} c_i r_i$ and assume γ is positive. Then the following statements are equivalent.

1. $F \in \mathrm{MDA}(H)$
2. $F^* \in \mathrm{MDA}(H)$

MDA of Conditional Layer Mixture Distributions

Let $F^{(k)}$ be a conditional layer mixture distribution with distributional components F_i and thresholds $0 = u_0 < u_1 < u_2 < \cdots$ for a positive integer $k > 1$. If $F_k \in MDA(H)$ for an extreme value distribution H, then $F^{(k)} \in MDA(H)$ again and the normalizing constants are

$$\begin{cases} a_n = \gamma^{1/\alpha} a_n^*, \quad b_n = 0, & \text{if } H \text{ is Fréchet,} \\ a_n = a_n^*, \quad b_n = b_n^* + a_n^* \log \gamma, & \text{if } H \text{ is Gumbel,} \\ a_n = \gamma^{-1/\alpha} a_n^*, \quad b_n = 0, & \text{if } H \text{ is Weibull.} \end{cases}$$

where a_n^* and b_n^* are the normalizing constants of F_k, $x_n^* = a_n^* x + b_n^*$, and

$$\gamma = \prod_{i=1}^{k-1} r_i \quad \text{where} \quad r_i = \frac{\overline{F_i}(u_i)}{\overline{F_{i+1}}(u_i)}.$$

Layer mixture is a linear mixture

Theorem The layer mixture distribution $F^{(k)}$ can be written as a linear mixture of the form

$$F^{(k)}(x) = c_1 H_1(x) + \sum_{j=1}^{k-1} \sum_{i=2j}^{2j+1} c_i H_{i-j}(x) I_{\{u_j \leq x\}}(x)$$

where $c_i = (-1)^{i-1}$ for $i = 1, 2, \cdots, 2k-1$ and $\overline{H_i}(x) = \overline{H_{i-1}}(u_{i-1})\overline{F_i}(x - u_{i-1})$ for $i = 2, 3, \cdots, k$ and $H_1 = F_1$.

4. Infinite Mixture and Hazard Rate Functions

Problems

- Threshold selection: Tail of $F^{(k)}$ is explained by F_k, the last distributional component only.
  - Can we model the tail by a mixture distribution again? (by infinitely many thresholds and distributional components)
- Smooth mixture distribution: Jumps of hazard rate functions at u_i lead to non-differentiable mixture distribution.

Hazard Rate Functions of Conditional Layer Mixture Distributions

Given distributions $\{F_i\}_{i=1,2,\ldots}$ and thresholds $0 = u_0 < u_1 < \ldots$, let h_i be the hazard function of F_i for each i. Then the conditional mixture distribution $F^{(k)}$ has the hazard function $h^{(k)}$ which is the mixture of the first k hazard functions, i.e.

$$\overline{F^{(k)}}(x) = \exp\Big(- \int_0^x h^{(k)}(s)\, ds\Big)$$

where $h^{(k)}$ is the simple mixture of the first k hazard functions such that

$$h^{(k)}(x) = \sum_{i=1}^{k-1} I_{\{u_{i-1} \le x < u_i\}}(x)\, h_i(x) + I_{\{u_{k-1} \le x\}}(x)\, h_k(x).$$

Hazard Rate Functions of Conditional Layer Mixture Distributions

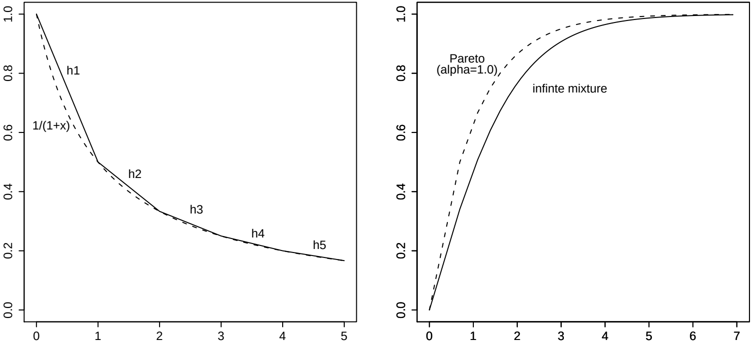

Figure: Mixture Hazard Function and Conditional Layer Mixture

Infinite Conditional Layer Mixture Distributions

Given distributions $\{F_i\}_{i=1,2,\ldots}$ and thresholds $0 = u_0 < u_1 < \ldots$, the infinite conditional layer mixture distribution denoted by $F^{(\infty)}$ is defined by

$$\overline{F^{(\infty)}}(x) = \sum_{i=1}^{\infty} I_{\{u_{i-1} \leq x < u_i\}}(x)\overline{F^{(i)}}(x)$$

or equivalently

$$\overline{F^{(\infty)}}(x) = \lim_{k\to\infty} \overline{F^{(k)}}(x)$$

where $F^{(k)}$ is the conditional layer mixture of the first k distributions.

Hazard Rate Functions of Infinite Mixture

Given distributions $\{F_i\}_{i=1,2,\ldots}$ and thresholds $0 = u_0 < u_1 < \ldots$, the infinite conditional layer mixture distribution $F^{(\infty)}$ has the mixture hazard function $h^{(\infty)}$, i.e.

$$\overline{F^{(\infty)}}(x) = \exp\Big(- \int_0^x h^{(\infty)}(s)\, ds\Big)$$

where $h^{(\infty)}$ is the simple mixture of h_i such that

$$h^{(\infty)}(x) = \sum_{i=1}^{\infty} I_{\{u_{i-1} \le x < u_i\}}(x) h_i(x),$$

Limiting Distribution of Infinite Mixture

Let $\delta = u_i - u_{i-1}$ for all i and consider a sequence of infinite mixture distributions $F^{(\infty)}(x;\delta)$. We are interested in the limiting distributions and limiting hazard functions as $\delta \to 0$ such as

$$\lim_{\delta \to 0} F^{(\infty)}(x) \quad \text{and} \quad \lim_{\delta \to 0} h^{(\infty)}(x).$$

Limiting Distribution of Infinite Mixture

Theorem Consider a sequence of thresholds $u_0 < u_1 < \ldots$ such that $\delta = u_i - u_{i-1} > 0$ for all i and a sequence of distributions $\{F_i(x;\delta)\}$ with hazard functions $\{h_i(x;\delta)\}$. If $\lim_{\delta\to 0} h^{(\infty)}(x;\delta)$ exists and $\lim_{\delta\to 0} F^{(\infty)}(x;\delta) < 1$, then the hazard function of the limiting distribution $F^{(\infty)}(x;\delta)$ is the limit of the mixture hazard function $h^{(\infty)}(x;\delta)$ as $\delta \to 0$, i.e.

$$\lim_{\delta\to 0} \overline{F^{(\infty)}}(x;\delta) = \exp\Big(- \int_0^x \lim_{\delta\to 0} h^{(\infty)}(s;\delta)\, ds\Big).$$

Moreover, if the limiting hazard function is continuous, the limiting distribution is differentiable.

Distributions of Limiting Hazard Functions

Corollary Any distribution F such that $\overline{F}(x) > 0$ with a continuous hazard rate function h is a limiting distribution of the infinite conditional mixture of exponential distributions, i.e.

$$\overline{F}(x) = \lim_{\delta \to 0} \overline{F^{(\infty)}}(x; \delta)$$

where $\overline{F}_i(x) = \exp(-h(u_i)x)$ for a sequence of thresholds $u_0 < u_1 < \cdots$ and $\delta = u_{i+1} - u_i$.

Distributions of Limiting Hazard Functions

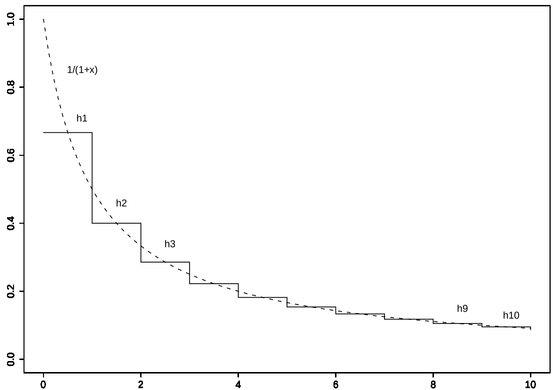

Figure: Pareto Hazard Function and Simple Function Approximation

5. Numerical Examples

Danish Fire Loss Data

d_i : number of losses in $[u_{i-1}, u_i)$
r_{i-1} : number of losses in $[u_{i-1}, \infty)$

Empirical hazard rate function can be written as

$$\hat{h}(x) = \frac{d_i}{(u_i - u_{i-1})(r_{i-1} - \frac{1}{2}d_i)}, \quad x \in [u_{i-1}, u_i)$$

for each positive integer i, which leads us to identify the components of the mixture hazard function by $\hat{h}$, i.e.

$$\hat{h}_i(x) = \hat{h}(u_{i-1}), \quad \text{for} \quad x \in [u_{i-1}, u_i)$$

Danish Fire Loss Data

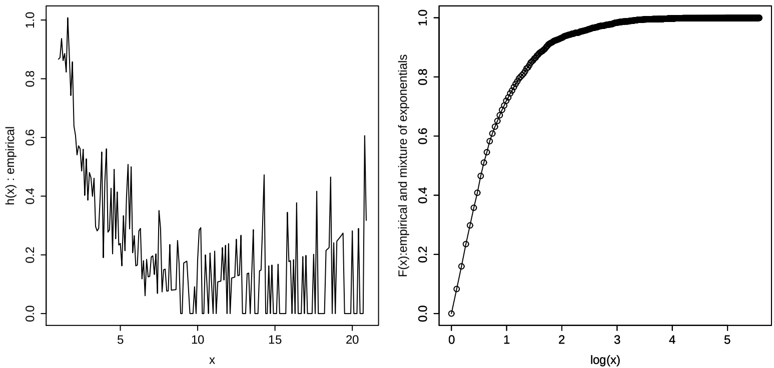

Figure: Empirical Hazard Function and Distribution Fit

g-and-h Distributions

The g-and-h distribution with four parameters (a, b, g, h) can be defined as a transformation of the standard normal random variable Z such that

$$F(x) = Pr(X \leq x), \quad X = a + b\,\frac{e^{gZ} - 1}{g} e^{\frac{hZ^2}{2}}$$

where g and h can be real valued functions of Z^2.

(K. Dutta and J. Perry, 2004)

g-and-h Distributions

$$(a, b, g, h) = (0, 1, 2.0, 0.2)$$

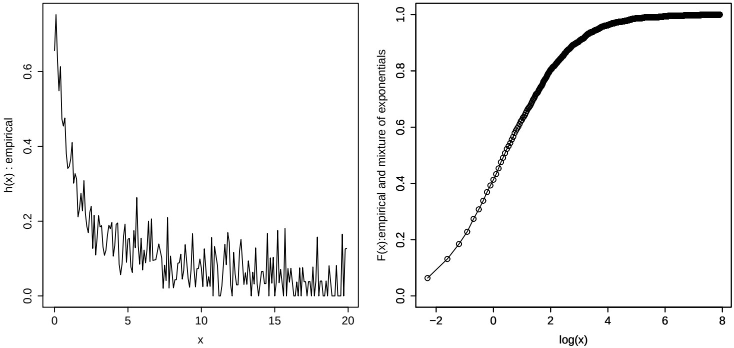

Figure: Empirical Hazard Function and Distribution Fit

Key References

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