09 Brownbag Seminar

# A generalization of Lucas polynomial sequence

G.-S.Cheon, Hana Kim and L.W. Shapiro

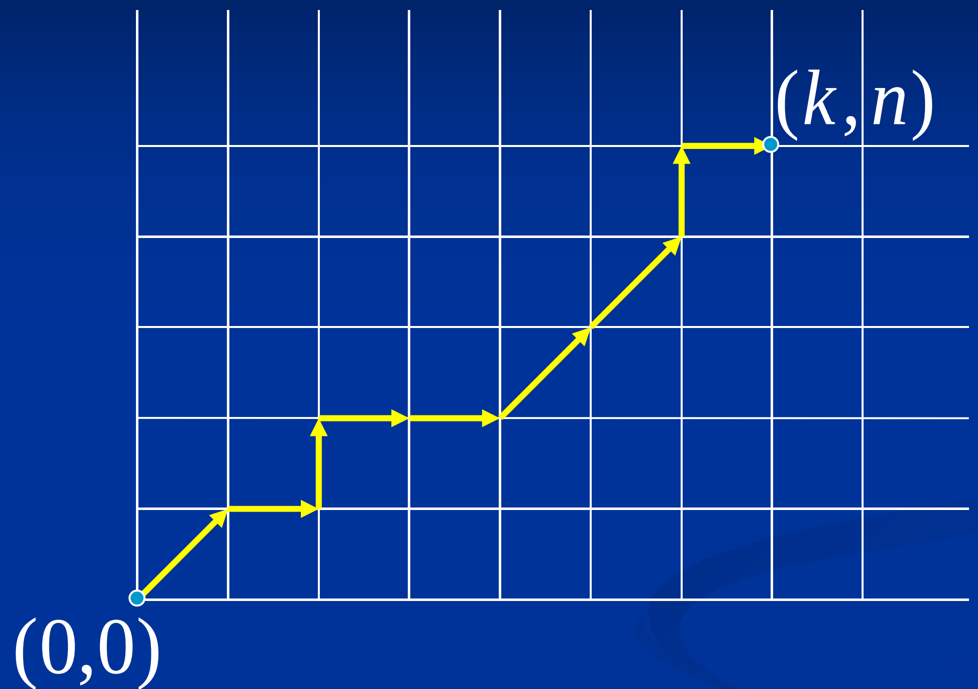
Sungkyunkwan University

2009. 10. 8

# Contents


- Delannoy numbers
- Weighted Delannoy numbers
- Riordan array $D_w(a, b, c)$
- Generalized Lucas polynomial sequence
- Combinatorial interpretations and examples

# Delannoy numbers




$$(1,0) = \rightarrow \quad (0,1) = \uparrow \quad (1,1) = \nearrow$$

$$D(n,k) = \sum_{d\geq 0} \binom{k}{d} \binom{n+k-d}{k} = \sum_{d\geq 0} \binom{k}{d} \binom{n}{d} 2^d$$

# Weighted Delannoy numbers

- $(a,b,c)$ - weighted path

$$(1,0) = \xrightarrow{a} \quad (0,1) = \uparrow_b \quad (1,1) = \overset{c}{\nearrow}$$

- The weight of a weighted path

- Theorem 1

  The total sum of the weights of all $(a,b,c)$-weighted paths from $(0,0)$ to $(k,n)$ on the lattice plane is

$$D_w(n,k) := \sum_{d\geq 0} \binom{k}{d} \binom{n+k-d}{k} a^{k-d} b^{n-d} c^{d}$$

- $D_w(n,k)$ : the weighted Delannoy numbers
- $D_w(n,k) = aD_w(n,k-1) + bD_w(n-1,k) + cD_w(n-1,k-1)$

  where $D_w(0,0)=1, D_w(0,n)=a^n, D_w(n,0)=b^n.$

# Riordan array $D_w(a,b,c)$

- L.W. Shapiro (1991)

  A Riordan array $\mathcal{R}(d_{n,k}) = (g(z), f(z))$

$$d_{n,k} = [z^n]g(z)(f(z))^k$$

  where

$$g(z) = 1 + g_1 z + g_2 z^2 + \ldots,$$
$$f(z) = z + f_2 z^2 + f_3 z^3 + \ldots$$

- Summation property (SP)

$$\sum_{k=0}^{n} d_{n,k} h_k = [z^n]g(z)h(f(z))$$

- $D_w(a,b,c) = [d_{n,k}]_{n,k\in N_0}$

**where** $d_{n,k} = \begin{cases} D_w(n-k,k) & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases}$

$$D_w(a,b,c) = \begin{bmatrix} 1 & & & & & \cdots \\ b & a & & & & \\ b^2 & c+2ab & a^2 & & & \\ b^3 & b(2c+3ab) & a(2c+3ab) & a^3 & & \\ b^4 & b^2(3c+4ab) & c^2+6abc+6b^2a^2 & a^2(3c+4ab) & a^4 & \\ & & \cdots & & & \end{bmatrix}$$

- **Theorem 2**

$D_w(a,b,c)$ is a Riordan array given by

$$D_w(a,b,c) = \left( \frac{1}{1-bz}, z\frac{a+cz}{1-bz} \right)$$

- **Lemma 3**

The generating function (GF) $\phi(z)$ for the row sums of $D_w(a,b,c)$ is given by

$$\phi(z) = \frac{1}{1-(a+b)z-cz^2}.$$

# Generalized Lucas polynomial sequence

- A.F. Horadam (1996)

  Polynomial sequence $\{W_n(x)\}$

$$W_n(x) = p(x)W_{n-1}(x) + q(x)W_{n-2}(x), \quad (n \geq 2) \qquad (*)$$

where

$$W_0(x) = c_0, W_1(x) = c_1 x^d, \; p(x) = c_2 x^d, q(x) = c_3 x^d$$

$d = 0$ or $1$.

- Lucas polynomial sequence of 1st kind $\{W_n(x)\}$

  If $c_0 = 1, W_1(x) = 1$ then

$$W_n(x) = \frac{u^n(x) - v^n(x)}{u(x) - v(x)},$$

where

$$u(x) + v(x) = p(x) \text{ and } u(x)v(x) = -q(x).$$

- Lucas polynomial sequence of 2nd kind $\{w_n(x)\}$

  If $c_0 = 2, W_1(x) = p(x)$ then

$$w_n(x) := W_n(x) = u^n(x) + v^n(x)$$

- Given $\overline{p}(x), \overline{q}(x) \in \mathrm{R}[x]$,
  by substituting $a = a(x), b = b(x)$ and $c = \overline{q}(x)$ into $D_w(a,b,c)$ such that $a + b = \overline{p}(x)$, we obtain a Riordan array $D_w(a,b,c) := D_w(\overline{p}(x), \overline{q}(x))$.

- Let $\overline{W_n}(x)$ be the $n$-th row sum of the Riordan array $D_w(\overline{p}(x), \overline{q}(x))$.

- $\displaystyle\sum_{n\geq 0} \overline{W_n}(x) z^n = \frac{1}{1 - \overline{p}(x)z - \overline{q}(x)z^2}.$

- Theorem 4

$$\overline{W}_n(x) = \sum_{k=0}^{\lceil (n-1)/2 \rceil} \binom{n-k}{k} (\overline{p}(x))^{n-2k} (\overline{q}(x))^k.$$

- Theorem 5

$$\overline{W}_n(x) = \overline{p}(x)\overline{W}_{n-1}(x) + \overline{q}(x)\overline{W}_{n-2}(x),$$

where $\overline{W}_0(x) = 1$ and $\overline{W}_1(x) = \overline{p}(x)$.

- $\overline{W}_n(x) = \dfrac{\overline{u}^n(x) - \overline{v}^n(x)}{\overline{u}(x) - \overline{v}(x)}$ where $\overline{u}(x) + \overline{v}(x) = \overline{p}(x)$ and $\overline{u}(x)\overline{v}(x) = -\overline{q}(x)$.

- Let $\overline{w}_n(x) := \overline{u}^n(x) + \overline{v}^n(x)$ and $\overline{w}_0(x) := \overline{p}(x)$. If $\overline{p}(x) = p(x)$ and $\overline{q}(x) = q(x)$ then $\overline{W}_n(x) = W_{n+1}(x)$ and $\overline{w}_n(x) = w_n(x)$ for $n = 0,1,2,\ldots$.

- We call $\{\overline{W}_n(x)\}$ and $\{\overline{w}_n(x)\}$ generalized Lucas polynomial sequences of the first kind and of the second kind, resp.

- **Theorem 6**

$$\overline{w}_n(x)=\sum_{k=0}^{\lceil n/2\rceil}\left\{\binom{n-k}{k-1}+\binom{n-k+1}{k}\right\}(\overline{p}(x))^{n-2k+1}(\overline{q}(x))^{k}.$$

- $$\sum_{n\geq 0}\overline{w}_{n-1}(x)z^n=\frac{1+\overline{q}(x)z^2}{1-\overline{p}(x)z-\overline{q}(x)z^2},\ (\overline{w}_{-1}:=1).$$

# Combinatorial interpretations and examples

- Theorem 7

  Let $\{\overline{W}_n(x)\}$ be the generalized Lucas polynomial sequence of the first kind. Then

$$\overline{W}_{n+1}(x)=\sum_{k=0}^{n}\omega_{k,n-k}(x) \qquad (n\geq 0)$$

where $\omega_{k,n-k}(x)$ is the sum of weights of $(a(x),b(x),c(x))$-weighted paths from $(0,0)$ to $(k,n-k)$ using the steps $(1,0),(0,1)$ and $(1,1)$ for which $a(x)+b(x)=\overline{p}(x)$ and $c(x)=\overline{q}(x)$.

- Special cases of the Lucas polynomial sequences $\{W_n(x)\}$

| $p(x)$ | $q(x)$ | $a(x)$ | $b(x)$ | $c(x)$ | $W_n(x)$ |
|---|---|---|---|---|---|
| $x$ | $1$ | $x$ | $0$ | $1$ | Fibonacci polynomials $F_n(x)$ |
| $2x$ | $1$ | $x$ | $x$ | $1$ | Pell polynomials $P_n(x)$ |
| $1$ | $2x$ | $1$ | $0$ | $2x$ | Jacobsthal polynomials $J_n(x)$ |
| $3x$ | $-2$ | $2x$ | $x$ | $-2$ | Fermat polynomials $\mathscr{F}_n(x)$ |
| $2x$ | $-1$ | $x$ | $x$ | $-1$ | Chebyshev polynomials of the 2nd kind $U_n(x)$ |

- **Example** (Fibonacci polynomial $F_n(x)$)
- the case of $a=x, b=0, c=1$; i.e, the Fibonacci polynomials are the row sums of the Riordan array $D_w(x,0,1)$:

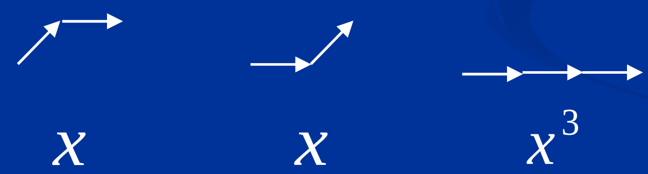
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 1 & x^2 & 0 & 0 \\ 0 & 0 & 2x & x^3 & 0 \\ 0 & 0 & 1 & 3x^2 & x^4 \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ 1+x^2 \\ 2x+x^3 \\ 1+3x^2+x^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ \vdots \end{bmatrix}$$

- the G.F. for $F_n(x) = \sum_{n\geq 0} F_n(x) z^n = \dfrac{z}{1-xz-z^2}$.

- Combinatorial interpretation for $F_n(x)$.

Consider $F_4(x) = 2x + x^3$. Since $F_4(x) = W_4(x) = \overline{W}_3(x)$ when $\overline{p}(x) = a(x) + b(x) = x$ and $\overline{q}(x) = c(x) = 1$, we may take $a(x) = x$, $b(x) = 0$ and $c(x) = 1$. Then

$$F_4(x) = \sum_{k=0}^{3} \omega_{k,3-k}(x).$$




$(x,0,1)$- weighted paths and their weights

- **Theorem 8**

  Let $\{\overline{w}_n(x)\}$ be the Lucas polynomial sequence of the second kind. Then

$$\overline{w}_n(x) = \sum_{k=0}^{n} \omega_{k,n-k}(x) + \overline{q}(x) \sum_{k=0}^{n-2} \omega_{k,n-k-2}(x). \qquad (n \geq 2)$$

- Special cases of the Lucas polynomial sequences $\{w_n(x)\}$

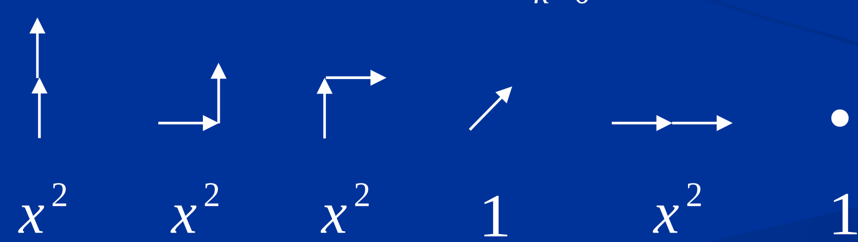
| $p(x)$ | $q(x)$ | $a(x)$ | $b(x)$ | $c(x)$ | $w_n(x)$ |
|---|---|---|---|---|---|
| $x$ | $1$ | $x$ | $0$ | $1$ | Lucas polynomials $L_n(x)$ |
| $2x$ | $1$ | $x$ | $x$ | $1$ | Pell-Lucas polynomials $Q_n(x)$ |
| $1$ | $2x$ | $1$ | $0$ | $2x$ | Jacobsthal-Lucas polynomials $j_n(x)$ |
| $3x$ | $-2$ | $2x$ | $x$ | $-2$ | Fermat-Lucas polynomials $f_n(x)$ |
| $2x$ | $-1$ | $x$ | $x$ | $-1$ | Chebyshev polynomials of the 1st kind $T_n(x)$ |

- **Example** (Pell-Lucas polynomial $Q_n(x)$)
- the G.F. for $Q_n(x) = \sum_{n\geq 0} Q_n(x) z^n = \dfrac{1+z^2}{1-2xz-z^2}, \quad (n \geq 1)$
- Combinatorial interpretation for $Q_n(x)$.

Consider $Q_2(x) = 2 + 4x^2$. Since $\bar{p}(x) = a(x) + b(x) = 2x$

and $\bar{q}(x) = c(x) = 1$, we may take $a(x) = x, b(x) = x$

and $c(x) = 1$. Then $Q_2(x) = \sum_{k=0}^{2} \omega_{k,2-k}(x) + 1 \cdot \omega_{0,0}(x)$.




$(x, x, 1)$- weighted paths and their weights

# Thank you for listening.