A characterization of the Riordan Bell subgroup by C-sequences

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A Riordan array $D = (g(z), f(z))$ is a infinite lower triangular matrix with k-th column generating function(GF) is $g(z)(f(z))^k$.

Example 1. (Pascal matrix)

$$\left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}$$

Theorem 1. *Let* $D = [d_{n,k}]_{n,k\geq 0}$ be a infinite lower triangular matrix. Then D is a Riordan array if and only if there exist an A-sequence $(a_0, a_1, a_2, \ldots)$ such that

$$d_{n+1,k+1} = \sum_{i\geq 0} a_i d_{n,k+i}. \tag{1}$$

Example 2. (Motzkin matrix)

$$(M, zM) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 \\ 9 & 12 & 9 & 4 & 1 & 0 \\ 21 & 30 & 25 & 14 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}$$

where $M = \frac{1-z-\sqrt{(1-z)^2-4z^2}}{2z^2}$.

Δ-sequence

Theorem 2. *Let* $D = (\frac{f(z)}{z}, f(z)) = [d_{n,k}]_{n,k\geq 0}$ be a Riordan array with every diagonal entries are 1. Then D is a pseudo-involution if and only if D has a Δ-sequence $(b_0, b_1, b_2, \ldots)$ such that

$$d_{n+1,k} = d_{n,k-1} + \sum_{i\geq 0} b_i d_{n-i,k+i}. \tag{2}$$

Example 3. (RNA triangle)

$$(G, zG) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 6 & 4 & 1 & 0 \\ 8 & 13 & 13 & 10 & 5 & 1 \\ & & \ldots & & & \end{bmatrix}$$

where $G = \frac{1-z+z^2-\sqrt{(1-z+z^2)^2-4z^2}}{2z^2}$.

Pseudo-involution

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 6 & 4 & 1 & 0 \\ 8 & 13 & 13 & 10 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ -2 & 3 & -3 & 1 & 0 & 0 \\ 4 & -6 & 6 & -4 & 1 & 0 \\ -8 & 13 & -13 & 10 & -5 & 1 \\ & & \cdots & & & \end{bmatrix}$$

Theorem 3. *Let* $D = (g(z), f(z)) = [d_{n,k}]_{n,k \geq 0}$ be a Riordan array. Then D has a form $(\frac{f(z)}{z}, f(z))$ if and only if D has a C-sequence $(c_0, c_1, c_2, \ldots)$ such that

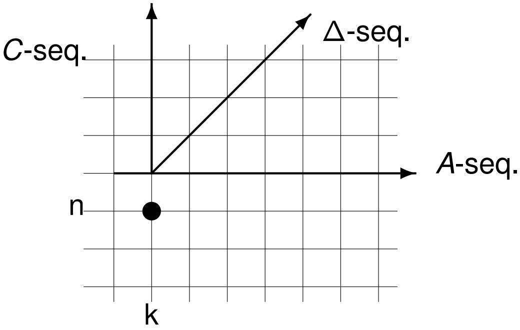
$$d_{n+1,k} = d_{n,k-1} + \sum_{i \geq 0} c_i d_{n-i,k}. \qquad (3)$$

Example 3. (Catalan matrix)

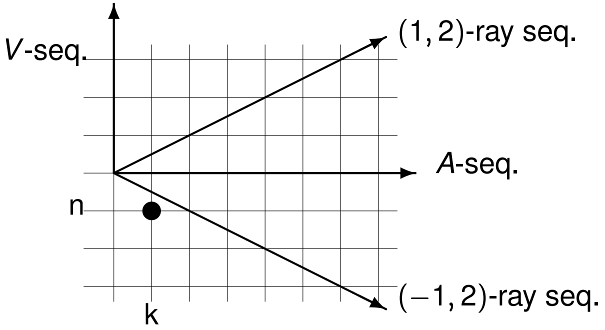
$$(C, zC) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 \\ 14 & 14 & 9 & 4 & 1 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}$$

where $C = \frac{1-\sqrt{1-4z}}{2z}$.

Sequences on the Lattice plane



Ray-Sequences



Note :

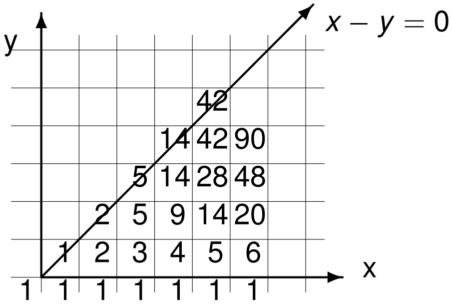
(i) V-sequence $(v_0, v_1, v_2, v_3, \ldots)$ such that $v_i = f_i$.

(ii) (a, b)-ray sequence $(s_0, s_1, s_2, s_3, \ldots)$ such that

$$d_{n+1,k+1} = \sum_{j \geq 0} s_j d_{n-aj,k+bj}$$

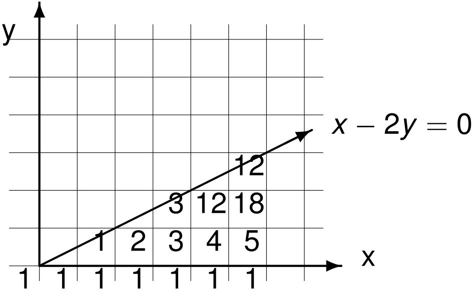
where $a + b > 0$ and $b > 0$.

Catalan Path



Note : The Catalan numbers ($C = 1 + zC^2$) counts the number of paths from $(0, 0)$ to (n, n) on the lattice plane $Z \times Z$ by using the steps $H = (1, 0)$, $U = (0, 1)$ where the paths cannot go above the line $x - y = 0$.

Ternary Path



Note : The Ternary numbers ($T = 1 + zT^3$) counts the number of paths from $(0, 0)$ to $(2n, n)$ on the lattice plane $Z \times Z$ by using the steps $H = (1, 0)$, $U = (0, 1)$ where the paths cannot go above the line $x - 2y = 0$.

Theorem 4. *Let* $Q^{(r)}$ be a quadratic equation such that $Q^{(r)} = a + bz(Q^{(r)})^r + cz(Q^{(r)})^{1+r} + dz^2(Q^{(r)})^{2r-1} + ez^2(Q^{(r)})^{2r}$. Then $[z^n]Q^{(r)}(z)$ counts the total sum of weight paths from $(0,0)$ to (rn,n) on the lattice plane $Z \times Z$ by using the steps $H = (1,0)$, $D = (1,1)$, $U = (0,1)$, $D_2 = (2,2)$ and $S = (1,2)$ with weights a, b, c, d and e, respectively, where the paths cannot go above the line $x - ry = 0$.