Local regularity of the Navier-Stokes equations near the curved boundary

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Outline

Navier-Stokes Equations

Navier-Stokes Equations

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla p = f, & \text{in} \quad Q_T = \Omega \times I \\ \operatorname{div} \mathrm{u} = 0 & \text{in} \quad Q_T = \Omega \times I \\ u = 0 & \text{on} \quad \partial\Omega \times I, \\ u(x,0) = u_0(x). & \end{cases}$$

Conditions

Zero-dimensional integral condition

$$\|v\|_{L^{p,q}_{x,t}(\Omega\times[0,T])} := \Big\| \|v(\cdot,t)\|_{L^p_x(\Omega)} \Big\|_{L^q_t(0,T)} < \infty, \qquad (1)$$

$$\frac{3}{p}+\frac{2}{q}\le 1, \quad 3\le p\le\infty,$$

(1) $\Longrightarrow$ weak solutions become regular in $\Omega\times[0,T)$.
G. Prodi (1959), T. Ohyama (1960), J. Serrin (1962), O. A. Ladyzhenskaya (1967), E. Fabes, B. Jones and N. Riviere (1972), H. Sohr (1983), Y. Giga (1986) , M. Struwe (1988), L. Escauriaza, G. Seregin and V. Šverák (2003)

- Caffarelli, Kohn and Nirenberg (1982)
  The one-dimensional parabolic Hausdorff measure of possible singular set is zero for suitable weak solutions. The key observation is the following regularity criterion :

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_{z,r}} |\nabla u(y,s)|^2 \, dyds \le \epsilon. \tag{2}$$

- G. A. Seregin (2002)
  The extension of (2) up to flat boundary.

- S. Gustafson, K. Kang & T.-P. Tsai (2006)
  There exist $\epsilon > 0$ such that if suitable weak solution u satisfies

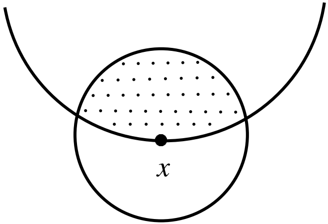
$$\limsup_{r\to 0} \frac{1}{r} \Big\| \|u\|_{L^p(B^+_{x,r})} \Big\|_{L^q(t-r^2,t)} \leq \epsilon,$$

where

$$\frac{3}{p} + \frac{2}{q} = 2, \quad 2 < q < \infty,$$

then u is regular in a neighborhood of $z = (x,t)$.

Notations



- For a point $x = (x', x_3) \in \mathbb{R}^3$ with $x' \in \mathbb{R}^2$
  $B_{x,r} = \{y \in \mathbb{R}^3 : |y - x| < r\}$,
  $D_{x',r} = \{y' \in \mathbb{R}^2 : |y' - x'| < r\}$.
- For $x \in \bar{\Omega}$, $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$.
  If $x = 0$, $\Omega_r := \Omega_{x,r}$.

Notations

- $W^{k,q}(\Omega) = \{u \in L^q(\Omega) \, : \, D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\}$.
- $⨍_E f := \frac{1}{|E|} \int_E f$.
- A solution u is said to be regular at $z = (x,t) \in \bar{\Omega} \times I$ if $u \in L^\infty(\Omega_{x,r} \times (t - r^2, t))$ for some $r > 0$ and z is called a regular point.
- Otherwise, u is singular at z and z is a singular point.
- $f \in M_{2,\gamma}(Q_T)$ for some $0 < \gamma \le 2$ means

$$\|f\|_{M_{2,\gamma}(Q_T)} = \sup_{r>0} \left\{ \left(\frac{1}{r^{1+2\gamma}} \int_{Q_{z,r}} |f|^2 \, dx \right)^{\frac{1}{2}} : z = (x,t) \in \bar{Q}_T \right\},$$

where $Q_{z,r} = (\Omega_{x,r} \times (t - r^2, t)) \cap Q_T$.

Assumption

Assumption 1

Suppose that Ω be a domain with C^2 boundary. For each point $x = (x', x_3) \in \partial\Omega$ there exist absolute constant L and r_0 independent of x such that we can find a Cartesian coordinate system $\{y_i\}_{i=1}^3$ with the origin at x and a C^2 function $\varphi : D_{r_0} \to \mathbb{R}$ satisfying

$$\Omega_{r_0} = \Omega \cap B_{x,r_0} = \{y = (y', y_3) \in B_{x,r_1} \,:\, y_3 > \varphi(y')\}$$

and

$$\varphi(0) = 0, \quad \nabla_y \varphi(0) = 0, \quad \sup_{D_{r_0}} \left|\nabla_y^2 \varphi\right| \le L.$$

Remark 1

- The main condition on the Assumption 1 is the uniform estimate of the C^2-norms of the function φ for each $x \in \partial\Omega$.
- There exists a sufficiently small r_1 with $r_1 < r_0$, where r_0 is the number in the Assumption 1 such that for any $r < r_1$

$$\sup_{x \in \partial\Omega} \|\varphi\|_{C^2(D_r)} \leq L(1 + r + r^2). \tag{3}$$

Theorem 1

Let u be a suitable weak solution of the (NSE) in Ω with force $f \in M_{2,\gamma}$ for some $\gamma > 0$. Assume that Ω is a bounded domain with C^2 boundary satisfying the Assumption 1. Suppose that $(x,t) \in \partial\Omega \times I$. For every pair p, q satisfying

$$1 \le \frac{3}{p} + \frac{2}{q} \le 2, \quad 2 < q \le \infty, \quad (p,q) \ne (\frac{3}{2}, \infty),$$

there exists $\epsilon > 0$ depending on p, q, γ and $\|f\|_{M_{2,\gamma}}$ such that if

$$\limsup_{r \to 0} r^{-(\frac{3}{p}+\frac{2}{q}-1)} \left\| \|u\|_{L^p(\Omega_{x,r})} \right\|_{L^q(t-r^2,t)} < \epsilon,$$

then u is regular at $z = (x,t)$.

Suitable Weak Solution

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain satisfying the Assumption 1 and $I = [0, T)$. We denote $Q_T = \Omega \times I$. Suppose that $f \in M_{2,\gamma}(Q_T)$ for some $\gamma > 0$. A pair of (u, p) is a suitable weak solution to (NSE) if the following conditions are satisfied:

(a) The functions $u : Q_T \to \mathbb{R}^3$ and $p : Q_T \to \mathbb{R}$ satisfy

$$u \in L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)), \quad p \in L^\lambda(I; L^{\kappa^*}),$$

$$\nabla^2 u \in L^\lambda(I; L^k(\Omega)), \quad \nabla p \in L^\lambda(I; L^\kappa),$$

where κ, κ^* and λ be numbers satisfying

$$\frac{3}{\kappa} + \frac{2}{\lambda} = 4, \quad \frac{1}{\kappa^*} = \frac{1}{\kappa} - \frac{1}{3}, \quad 1 < \lambda < 2.$$

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Suitable Weak Solution

(b) u and p solve the (NSE) in Q_T in the sense of distributions and u satisfies the boundary condition $u = 0$ on $\partial\Omega \times I$.

(c) u and p satisfy the local energy inequality

$$\int_\Omega |u(x,t)|^2 \phi(x,t)dx + 2\int_{t_0}^t \int_\Omega |\nabla u(x,t')|^2 \phi(x,t')dz$$

$$\leq \int_{t_0}^t \int_\Omega \Big(|u|^2 (\partial_t \phi + \Delta\phi) + \Big(|u|^2 + 2p\Big) u \cdot \nabla\phi + 2f \cdot u\phi \Big)\, dz,$$

for all $t \in I = (0,T)$ and for all non-negative functions $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$, vanishing in a neighborhood of the set $\Omega \times \{t = 0\}$.

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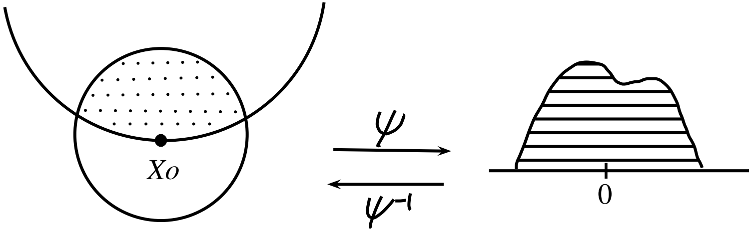

- Flatting the boundary near x_0,

$$x = \psi(y) \equiv (y_1,\, y_2,\, y_3 - \varphi(y_1, y_2))$$

- Define

$$v = u \circ \psi^{-1}, \quad \pi = p \circ \psi^{-1}, \quad g = f \circ \psi^{-1} \quad \text{in } \psi(\Omega_{x_0,\rho}).$$

Perturbed Navier-Stokes equations

$$\begin{cases} v_t - \hat{\Delta} v + (v \cdot \hat{\nabla}) v + \hat{\nabla} \pi = g, & \text{in} \quad \psi(\Omega_{x_0,\rho}) \\ \hat{\nabla} \cdot v = 0 & \text{in} \quad \psi(\Omega_{x_0,\rho}) \\ v = 0 & \text{on} \quad \partial\psi(\Omega_{x_0,\rho}) \cap \{x_3 = 0\}, \end{cases} \tag{4}$$

Differential Operators

$$\hat{\nabla} = (\partial_{x_1} - \varphi_{x_1}\partial_{x_3}, \partial_{x_2} - \varphi_{x_2}\partial_{x_3}, \partial_{x_3})$$
$$\hat{\Delta} = a_{ij}(x)\partial^2_{x_i,x_j} + b_i(x)\partial_{x_i}, \tag{5}$$

where a_{ij} and b_i are given as

$$a_{ij}(x) = \delta_{ij}, \quad a_{i3}(x) = a_{3i}(x) = -\varphi_{x_i}, \quad b_i(x) = 0 \qquad i = 1, 2,$$

and

$$a_{33}(x) = 1 + \sum_{i=1}^{2}(\varphi_{x_i})^2, \qquad b_3(x) = -\sum_{i=1}^{2}\varphi_{x_i x_i}.$$

Remark

$$\frac{1}{2}\left|\nabla v(x,t)\right| \leq \left|\hat{\nabla} v(x,t)\right| \leq 2\left|\nabla v(x,t)\right| \qquad \text{for all } x \in \psi(\Omega_{(x_0),2r}),$$

$$B^{+}_{\psi(x_0),\frac{r}{2}} \subset \psi(\Omega_{x_0,r}) \subset B^{+}_{\psi(x_0),2r},$$

Remark

$$\int_{\psi(\Omega_{r_0})} \left|v(x,t)\right|^2 \xi(x,t)dx + 2\int_{t_0}^{t}\int_{\psi(\Omega_{r_0})} \left|\hat{\nabla} v(x,t')\right|^2 \xi(x,t')dz$$

$$\leq \int_{t_0}^{t}\int_{\psi(\Omega_{r_0})} \left(\left|v\right|^2 (\partial_t \xi + \hat{\Delta}\xi) + \left(\left|v\right|^2 + 2\pi\right) u \cdot \hat{\nabla}\xi + 2f \cdot v\xi\right) dz,$$

Scaling invariant functionals

For any $r < r_1$ and a suitable weak solution (u, p) of (NSE),

$$A(r) := \frac{1}{r^2}\int_{\Omega_r} |u(y,s)|^3\, dyds, \quad E(r) := \frac{1}{r}\int_{Q_r} |\nabla u(y,s)|^2\, dyds,$$

$$D(r) := \sup_{-r^2 \le s \le 0} \frac{1}{r}\int_{\Omega_r} |u(y,s)|^2\, dy,$$

$$K(r) := \frac{1}{r}\left(\int_{t-r^2}^{t}\left(\int_{\Omega_r} |u(y,s)|^p\right)^{\frac{q}{p}} ds\right)^{\frac{1}{q}},$$

$$S(r) := \frac{1}{r}\left(\int_{t-r^2}^{t}\left(\int_{\Omega_r} |p(y,s)|^{\kappa^*}\, dy\right)^{\frac{\lambda}{\kappa^*}} ds\right)^{\frac{1}{\lambda}},$$

where p, q, κ, κ^* and λ be numbers satisfying

$$\frac{3}{\kappa} + \frac{2}{\lambda} = 4,\ \frac{1}{\kappa^*} = \frac{1}{\kappa} - \frac{1}{3},\ \frac{1}{p} + \frac{1}{\kappa^*} = 1,\ \frac{1}{q} + \frac{1}{\lambda} = 1,\ 1 < \lambda < 2. \quad (6)$$

Scaling invariant functionals

For a weak solution (v, π) and $B_r^+ \subset \psi(\Omega_{r_1})$,

$$\hat{A}(r) := \frac{1}{r^2} \int_{Q_r^+} |v(y,s)|^3 \, dyds, \quad \hat{E}(r) := \frac{1}{r} \int_{Q_r^+} |\hat{\nabla} v(y,s)|^2 \, dyds,$$

$$\hat{D}(r) := \sup_{-r^2 \le s \le 0} \frac{1}{r} \int_{B_r^+} |v(y,s)|^2 \, dy,$$

$$\hat{K}(r) := \frac{1}{r} \left(\int_{t-r^2}^{t} \left(\int_{B_r^+} |v(y,s)|^p \, dy \right)^{\frac{q}{p}} ds \right)^{\frac{1}{q}},$$

Scaling invariant functionals

$$\hat{S}(r) := \frac{1}{r}\left(\int_{t-r^2}^{t}\left(\int_{B_r^+}|\pi(y,s)|^{\kappa^*}dy\right)^{\frac{\lambda}{\kappa^*}}ds\right)^{\frac{1}{\lambda}},$$

$$\hat{S}_1(r) := \frac{1}{r}\left(\int_{t-r^2}^{t}\left(\int_{B_r^+}|\nabla\pi(y,s)|^{\kappa}dy\right)^{\frac{\lambda}{\kappa}}ds\right)^{\frac{1}{\lambda}},$$

$$\hat{S}_a(r) := \frac{1}{r}\left(\int_{t-r^2}^{t}\left(\int_{B_r^+}|\pi(y,s)-(\pi)_{B_r^+}(s)|^{\kappa^*}dy\right)^{\frac{\lambda}{\kappa^*}}ds\right)^{\frac{1}{\lambda}},$$

where $(\pi)_{B_r^+} = \fint_{B_r^+}\pi(y,s)dy$.

Remark 2

Let $x = \psi(x_0)$. Then there exist sufficiently small r_1 and an absolute constant C such that for any $4r < r_1$ the followings are satisfied:

$$\frac{1}{C}E(r) \le \hat{E}(2r) \le CE(4r), \quad \frac{1}{C}A(r) \le \hat{A}(2r) \le CA(4r),$$

$$\frac{1}{C}K(r) \le \hat{K}(2r) \le CK(4r), \quad \frac{1}{C}S(r) \le \hat{S}(2r) \le CS(4r),$$

$$\frac{1}{C}D(r) \le \hat{D}(2r) \le CD(4r),$$

$$\|g\|_{M^{2,\gamma}(\Pi_r)} \le C\,\|f\|_{M^{2,\gamma}(Q_r)}.$$

Lemma 1

Let Ω be a bounded domain satisfying the Assumption 1 and $x_0 \in \partial\Omega$. Suppose that (v, π) is a suitable weak solution of (4) in $\psi(\Omega_{x_0}) \subset \mathbb{R}^3_+$. Let $z = (x, t)$ with $x = \psi(x_0)$. Assume $g \in M_{2,\gamma}$ for some $\gamma \in (0, 2]$. Then there exist $\epsilon > 0$ and r_1 depending on $\gamma, \|g\|_{M_{2,\gamma}}$ such that if

$$\hat{A}^{\frac{1}{3}}(r) + \hat{S}_a(r) < \epsilon \quad \text{for some } r < r_1,$$

then z is a regular point.

The proof of the Lemma 1 is based on the decay property of the following scaling invariant functionals:

Lemma 2

Let $0 < \theta < \frac{1}{2}$ and $\beta \in (0, \gamma)$. Under the same assumption as in the Lemma 1, there exist $\epsilon_1 > 0$ and r_1 depending on θ, γ, β and m_γ such that if $\hat{A}^{\frac{1}{3}}(r) + \hat{S}_a(r) + m_\gamma r^{\beta+1} < \epsilon_1$ for some $r \in (0, r_1)$, then

$$\hat{A}^{\frac{1}{3}}(\theta r) + \hat{S}_a(\theta r) < C\theta^{1+\alpha} \left(\hat{A}^{\frac{1}{3}}(r) + \hat{S}_a(r) + m_\gamma r^{\beta+1} \right),$$

where $0 < \alpha < 1$ and C is a constant.

The sketch of the proof of the Lemma 2

- Denote $\phi(r) := \hat{A}^{\frac{1}{3}}(r) + \hat{S}_a(r) + m_\gamma r^{\beta+1}$.

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- Denote $\phi(r) := \hat{A}^{\frac{1}{3}}(r) + \hat{S}_a(r) + m_\gamma r^{\beta+1}$.
- Suppose not. Then for any $\alpha \in (0,1)$ and $C > 0$, there exist $z_n = (x_n, t_n), r_n \searrow 0$ and $\epsilon_n \searrow 0$ such that

$$\phi(r_n) = \epsilon_n, \qquad \hat{A}^{\frac{1}{3}}(\theta r_n) + \hat{S}_a(\theta r_n) > C\theta^{1+\alpha}\phi(r_n) = C\theta^{1+\alpha}\epsilon_n.$$

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- Let $w = (y, s)$ where $y = \frac{1}{r_n}(x - x_n)$, $s = \frac{1}{r_n^2}(t - t_n)$. Define $\hat{v}_n$, $\hat{\pi}_n$ and $\hat{g}_n$ by

$$\hat{v}_n(w) = \frac{1}{\epsilon_n} r_n v_n(z), \quad \hat{\pi}_n(w) = \frac{1}{\epsilon_n} r_n^2 \pi_n(z),$$

$$\hat{g}_n(w) = \frac{1}{\epsilon_n} r_n^3 g_n(z).$$

The sketch of the proof of the Lemma2

- Due to scaling and change of variables,

$$\hat{A}(\hat{v}_n,\theta) := \frac{1}{\theta^2}\int_{Q_\theta^+} |\hat{v}_n|^3 \, dw,$$

$$\hat{S}_1(\hat{\pi}_n,\theta) := \frac{1}{\theta}\left(\int_{-\theta^2}^0 \left(\int_{B_\theta^+} |\hat{\nabla}\hat{\pi}_n|^\kappa \, dy\right)^{\frac{\lambda}{\kappa}} ds\right)^{\frac{1}{\lambda}},$$

$$\hat{S}_a(\hat{\pi}_n,\theta) := \frac{1}{\theta}\left(\int_{-\theta^2}^0 \left(\int_{B_\theta^+} |\hat{\pi}_n - (\hat{\pi}_n)_{B_\theta^+}|^{\kappa^*} \, dy\right)^{\frac{\lambda}{\kappa^*}} ds\right)^{\frac{1}{\lambda}},$$

$$\|\hat{v}_n\|_{L^3(Q_1^+)} + \|\hat{\pi}_n\|_{L^{\kappa^*,\lambda}(Q_1^+)} + m_\gamma^n r_n^{\beta+1} = 1,$$

$$\hat{A}^{\frac{1}{3}}(\hat{v}_n,\theta) + \hat{S}_a(\hat{\pi}_n,\theta) \geq C\theta^{1+\alpha}.$$

The sketch of the proof of the Lemma2

- Weak convergence of $\hat{v}_n$ and $\hat{\pi}_n$

$$\hat{v}_n \rightharpoonup \hat{v} \quad \text{in } L^3(Q_1^+),$$

$$\hat{\pi}_n \rightharpoonup \hat{\pi} \quad \text{in } L^{\kappa^*,\lambda}(Q_1^+), \quad (\hat{\pi})_{B_1^+}(s) = 0.$$

$$\partial_s \hat{v}_n \rightharpoonup \partial_s \hat{v} \quad \text{in } L^\lambda((-1,0);(w^{2,2}(B_1^+))').$$

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$$\partial_s \hat{v}_n \rightharpoonup \partial_s \hat{v} \quad \text{in } L^\lambda((-1,0);(w^{2,2}(B_1^+))').$$

- By the local energy inequality,

$$\hat{\nabla} \hat{v}_n \rightharpoonup \hat{\nabla} \hat{v} \quad \text{in } L^2(Q_{3/4}^+), \qquad \hat{v}_n \rightharpoonup \hat{v} \quad \text{in } W^{1,2}(Q_{3/4}^+).$$

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- Due to the local boundary estimate of the Stokes system,

$$\partial_s \hat{v}_n, \hat{\Delta} \hat{v}_n, \hat{\nabla} \hat{\pi}_n \rightharpoonup \partial_s \hat{v}, \hat{\Delta} \hat{v}, \hat{\nabla} \hat{\pi} \quad \text{in } L^{\kappa,\lambda}(Q_{5/8}^+).$$

The sketch of the proof of the Lemma2

- Strong convergence of $\hat{v}_n$ in L^3-norm

$$\hat{A}(\hat{v}_n,\theta) \to \hat{A}(\hat{v},\theta), \qquad \hat{A}^{\frac{1}{3}}(\hat{v},\theta) \le C_1\theta^{1+\alpha_0},$$

The sketch of the proof of the Lemma2

- Strong convergence of $\hat{v}_n$ in L^3-norm

$$\hat{A}(\hat{v}_n,\theta)\to\hat{A}(\hat{v},\theta),\qquad \hat{A}^{\frac{1}{3}}(\hat{v},\theta)\le C_1\theta^{1+\alpha_0},$$

- Decomposition ($\hat{v}_n=\bar{v}_n+\tilde{v}_n$ and $\hat{\pi}_n=\bar{\pi}_n+\tilde{\pi}_n$)

$$\partial_s\bar{v}_n-\hat{\Delta}\bar{v}_n+\hat{\nabla}\bar{\pi}_n=-\epsilon_n(\bar{v}_n\cdot\hat{\nabla})\bar{v}_n+\hat{g}_n,\qquad \mathrm{div}\bar{v}_n=0\quad\text{in }\tilde{Q}^+,$$

$$(\bar{\pi}_n)_{\tilde{B}^+}(s)=0,\quad s\in\Big(-\Big(\frac{3}{4}\Big)^2,0\Big),$$

$$\bar{v}_n=0\quad\partial\tilde{B}^+\times\Big[-\Big(\frac{3}{4}\Big)^2,0\Big],\qquad \bar{v}_n=0\quad\tilde{B}^+\times\Big\{s=-\Big(\frac{3}{4}\Big)^2\Big\}.$$

The sketch of the proof of the Lemma2

- Equation of $(\tilde{v}_n, \tilde{\pi}_n)$

$$\partial_s \tilde{v}_n - \hat{\Delta} \tilde{v}_n + \hat{\nabla} \tilde{\pi}_n = 0, \quad \text{div } \tilde{v}_n = 0 \quad \text{in } \tilde{Q}^+,$$

$$\tilde{v}_n = 0 \quad \text{on } (\tilde{B}^+ \cap \{x_3 = 0\}) \times \Big[-\Big(\frac{3}{4}\Big)^2, 0 \Big].$$

The sketch of the proof of the Lemma2

- Equation of $(\tilde{v}_n, \tilde{\pi}_n)$

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$$\tilde{v}_n = 0 \quad \text{on } (\tilde{B}^+ \cap \{x_3 = 0\}) \times \Big[-\Big(\frac{3}{4}\Big)^2, 0 \Big].$$

- For $3/\tilde{\kappa} + 2/\lambda = 1$,

$$\|\hat{\nabla}^2 \tilde{v}_n\|_{L^{\tilde{\kappa},\lambda}(Q^+_{16/9})} + \|\hat{\nabla}\tilde{\pi}_n\|_{L^{\tilde{\kappa},\lambda}(Q^+_{9/16})} \le C(1+\epsilon_n).$$

The sketch of the proof of the Lemma2

- By the Poincaré inequality,

$$\hat{S}_a(\hat{\pi}_n, \theta) \leq C_2\Big(\hat{S}_1(\bar{\pi}_n, \theta) + \hat{S}_1(\tilde{\pi}_n, \theta)\Big),$$

□

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- Decay of $\tilde{\pi}_n$

$$\hat{S}_1(\tilde{\pi}_n, \theta) \le C\theta^2(1+\epsilon_n), \quad \hat{S}_1(\bar{\pi}_n, \theta) \le C\epsilon_n.$$

$$\liminf_{n\to\infty} \hat{S}_a(\hat{\pi}_n, \theta) \le \lim_{n\to\infty} C_2\theta^2(1+\epsilon_n) \le C_2\theta^{1+\alpha_0}$$

□

The sketch of the proof of the Lemma2

- By the Poincaré inequality,

$$\hat{S}_a(\hat{\pi}_n,\theta) \le C_2\Big(\hat{S}_1(\bar{\pi}_n,\theta) + \hat{S}_1(\tilde{\pi}_n,\theta)\Big),$$

- Decay of $\tilde{\pi}_n$

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- Contradiction

$$2(C_1+C_2)\theta^{1+\alpha_0} \le C\theta^{1+\alpha_0}$$

$$\le \liminf_{n\to\infty}(\hat{A}^{\frac{1}{3}}(\hat{v}_n,\theta) + \hat{S}_a(\hat{\pi}_n,\theta)) \le (C_1+C_2)\theta^{1+\alpha_0}.$$

□

- (Pressure estimate)
  Suppose $w = (y, s), y \in \Gamma, t - \rho^2 > 0$, and $t < T$. Then for $0 \le r \le \rho/4$,

$$\hat{S}_1(r) \le C\Big(\Big(\frac{\rho}{r}\Big)\big(\hat{E}^{\frac{1}{\lambda}}(\rho)\hat{D}^{\frac{3-2\kappa}{2\kappa}}(\rho) + \rho^{\gamma+1}m_\gamma\big) + \Big(\frac{r}{\rho}\Big)\big(\hat{E}^{\frac{1}{2}}(\rho) + \hat{S}_1(\rho)\big)\Big),$$

  where κ and λ are numbers in (6).

The sketch of the proof of the Main Theorem

- By Pressure estimate,

$$\hat{A}(r)+\hat{S}_1(r)\le C\hat{D}^{\frac{1}{q}}(r)\hat{E}^{1-\frac{1}{q}}(r)\hat{K}(r)$$

$$+C\left(\frac{\rho}{r}\right)\left(\hat{E}^{\frac{1}{\lambda}}(\rho)\hat{D}^{\frac{3-2\kappa}{2\kappa}}(\rho)+\rho^{\gamma+1}m_\gamma\right)+C\left(\frac{r}{\rho}\right)\left(\hat{E}^{\frac{1}{2}}(\rho)+\hat{S}_1(\rho)\right).$$

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- Choose small $\theta\in[0,1/2]$. Replacing r,ρ by θr and r,

$$\hat{A}(\theta r)+\hat{S}_1(\theta r)\le C\Bigg(\Big(\frac{1}{\theta^3}\hat{K}(r)+\frac{1}{\theta}\hat{K}^{\frac{1}{2}}(r)+\theta\Big)\hat{A}(r)$$

$$+\Big(\frac{1}{\theta^3}\hat{K}(r)+\frac{1}{\theta}\hat{K}^2(r)+\theta\Big)\hat{S}_1(r)+\mu(r)\Bigg),$$

where

$$\mu(r)=\frac{1}{\theta}\left(\hat{K}^{\frac{3}{7}}(r)+\hat{K}^2(r)\right)+\frac{1}{\theta}m_\gamma r^{\gamma+1}+\frac{1}{\theta}m_\gamma^2 r^{2(\gamma+1)}\hat{K}(r)$$

$$+\frac{1}{\theta}m_\gamma^2 r^{2(\gamma+1)}+m_\gamma^{\frac{2}{3}}r^{\frac{2}{3}(\gamma+1)}\hat{K}^{\frac{1}{3}}(r)+m_\gamma^{\frac{4}{3}}r^{\frac{4}{3}(\gamma+1)}\hat{K}^{\frac{2}{3}}(r).$$

The sketch of the proof of the Main Theorem

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$$\hat{A}(\theta^k r) + \hat{S}_1(\theta^k r) \leq \left(\frac{1}{2}\right)^k (\hat{A}(r) + \hat{S}_1(r)) + \frac{\epsilon}{4}.$$

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- For a sufficiently small $r_2 < r_1$

$$\hat{A}(r_2) + \hat{S}_a(r_2) \leq \hat{A}(r_2) + C_d\hat{S}_1(r_2) \leq \frac{\epsilon}{2},$$

□