An orthogonality between pairs of split decompositions for a Q-polynomial distance-regular graph and its application

Joohyung Kim

First consider a tridiagonal pair for which the underlying vector space supports a positive definite Hermitian form. I will show the duality between pairs of split decompositions for an underlying vector space of a tridiagonal pair with respect to a positive definite Hermitian form.

Next by using the above result and some results about the Terwilliger algebra, I will show a duality between pairs of split decompositions of the standard module for a Q-polynomial distance-regular graph with respect to the standard Hermitian form. For this decomposition we compute the complex conjugate and transpose of the associated primitive idempotents.

A tridiagonal pair is defined as follows.

V will denote a vector space over $\mathbb{C}$ with finite positive dimension.

We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$.

We say the pair A, A^* is a **tridiagonal pair** (or **TD pair**) on V whenever 1–4 hold below.

1. A and A^* are both diagonalizable on V.
2. There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \qquad (0 \leq i \leq d),$$

   where $V_{-1} = 0, V_{d+1} = 0$.
3. There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \qquad (0 \leq i \leq \delta),$$

   where $V_{-1}^* = 0,\ V_{\delta+1}^* = 0$.
4. There is no subspace W of V such that both $AW \subseteq W$, $A^* W \subseteq W$, other than $W = 0$ and $W = V$.

Referring to Definition of a tridiagonal pair, it turns out that

$d = \delta;$

we call this the **diameter** of the pair.

We now recall four direct sum decompositions of V called the split decompositions.

Lemma With reference to Definition of a tridiagonal pair, for $\mu, \nu \in \{\downarrow, \uparrow\}$ we have

$$V = \sum_{i=0}^{d} U_i^{\mu\nu} \qquad \text{(direct sum)},$$

where

$$\begin{aligned}
U_i^{\downarrow\downarrow} &= (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i}),\\
U_i^{\uparrow\downarrow} &= (V_{d-i}^* + \cdots + V_d^*) \cap (V_0 + \cdots + V_{d-i}),\\
U_i^{\downarrow\uparrow} &= (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d),\\
U_i^{\uparrow\uparrow} &= (V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d).
\end{aligned}$$

Let A, A^* denote a tridiagonal pair on V as in Definition of a tridiagonal pair.

For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*.)

We assume that there exists a positive definite Hermitian form $(\ ,\ )$ on V satisfying

$$
\begin{aligned}
(Au, v) &= (u, Av) \qquad u, v \in V, &(1)\\
(A^*u, v) &= (u, A^*v) \qquad u, v \in V. &(2)
\end{aligned}
$$

With reference to Assumption, the following 1,2 hold.

1. The eigenspaces $V_0, V_1, \ldots, V_d$ are mutually orthogonal with respect to $(\, , \,)$.
2. The eigenspaces $V_0^*, V_1^*, \ldots, V_d^*$ are mutually orthogonal with respect to $(\, , \,)$.

With reference to Assumption, the following 1,2 hold for $0 \leq i, j \leq d$ such that $i + j \neq d$.

1. The subspaces $U_i^{\downarrow\downarrow}$ and $U_j^{\uparrow\uparrow}$ are orthogonal with respect to $(\, , \,)$.
2. The subspaces $U_i^{\downarrow\uparrow}$ and $U_j^{\uparrow\downarrow}$ are orthogonal with respect to $(\, , \,)$.

Let X denote a nonempty finite set.

Let $V = \mathbb{C}X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$.

We call V the **standard module**.

Let V be the standard module.

We endow V with the Hermitian form $\langle\, ,\, \rangle$ that satisfies

$\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$.

Observe that $\langle\, ,\, \rangle$ is positive definite.

We call this form the **standard Hermitian form** on V.

Observe that for $B \in \mathrm{Mat}_X(\mathbb{C})$,

$$\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle \qquad\qquad u, v \in V. \tag{3}$$

Let $\Gamma = (X, R)$ denote a graph.

Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the **diameter** of Γ.

We say that Γ is **distance-regular** whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of x and y.

The p^h_{ij} are called the **intersection numbers** of Γ.

For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the ith **distance matrix** of Γ.

Observe that distance matrices of Γ are real symmetric.

We abbreviate $A := A_1$ and call this the **adjacency matrix** of Γ.

We find $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$.

We call M the **Bose-Mesner algebra** of Γ.

It turns out that A generates M.

By (3) and since A is real symmetric,

$$\langle Au, v\rangle = \langle u, Av\rangle \qquad u, v \in V. \tag{4}$$

M has a second basis $E_0, E_1, \ldots, E_D$ such that

1. $E_0 = |X|^{-1}J$,
2. $\sum_{i=0}^{D} E_i = I$,
3. $\overline{E_i} = E_i \ (0 \le i \le D)$,
4. $E_i^t = E_i \ (0 \le i \le D)$,
5. $E_i E_j = \delta_{ij} E_i \ (0 \le i, j \le D)$.

We call $E_0, \ldots, E_D$ the **primitive idempotents** of Γ.

Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$.

Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, so M is closed under $\circ$.

Thus there exist complex scalars q^h_{ij} $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \leq i, j \leq D).$$

It turns out that q^h_{ij} is real and nonnegative for $0 \leq h, i, j \leq D$.

The q^h_{ij} are called the **Krein parameters**.

The graph Γ is said to be Q**-polynomial** with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) if one of h, i, j is greater than (resp. equal to) the sum of the other two.

Fix a vertex $x \in X$.

For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x,y) = i \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad (y \in X). \tag{5}$$

We call E_i^* the ith **dual idempotent** of Γ with respect to x.

Observe that $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$.

We call M^* the **dual Bose-Mesner algebra** of Γ with respect to x.

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix with yy entry $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$.

Then $A_0^*, A_1^*, \ldots, A_D^*$ form a basis for M^*.

We call $A_0^*, A_1^*, \ldots, A_D^*$ the **dual distance matrices** of Γ with respect to x.

Observe that dual distance matrices of Γ are real symmetric.

We abbreviate $A^* := A^*_1$ and call this the **dual adjacency matrix** of Γ with respect to x.

The matrix A^* generates M^*.

By (3) and since A^* is real symmetric,

$$\langle A^* u, v \rangle = \langle u, A^* v \rangle \qquad\qquad u, v \in V. \qquad (6)$$

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

We call T the **subconstituent algebra** (or **Terwilliger algebra**) of Γ with respect to x.

We observe that T is generated by A, A^*.

We assume that $\Gamma = (X, R)$ is a distance-regular graph with diameter $D \geq 3$.

We assume that Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents.

We fix $x \in X$ and write $A^* = A^*(x)$, $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $T = T(x)$.

We abbreviate $V = \mathbb{C}X$.

For notational convenience we define $E_{-1} = 0$, $E_{D+1} = 0$ and $E_{-1}^* = 0$, $E_{D+1}^* = 0$.

Referring to Notational convention, by a **T-module** we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$.

Let W denote a T-module. Then W is said to be **irreducible** whenever W is nonzero and W contains no T-modules other than 0 and W.

Observe that each T-module is an orthogonal direct sum of irreducible T-modules.

In particular V is an orthogonal direct sum of irreducible T-modules.

Let W denote an irreducible T-module.

By the **endpoint** of W we mean $\min\{i | 0 \leq i \leq D,\ E_i^* W \neq 0\}$.

By the **diameter** of W we mean $|\{i | 0 \leq i \leq D,\ E_i^* W \neq 0\}| - 1$.

By the **dual endpoint** of W we mean

$\min\{i | 0 \leq i \leq D,\ E_i W \neq 0\}$.

With reference to Notational convention, let W denote an irreducible T-module.

Then A and A^* act on W as a tridiagonal pair in the sense of Definition of a tridiagonal pair.

Let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then for $\mu, \nu \in \{\downarrow, \uparrow\}$ we have

$$W = \sum_{h=0}^{d} W_h^{\mu\nu} \qquad \text{(direct sum)}, \tag{7}$$

where for $0 \leq h \leq d$,

$$\begin{aligned}
W_h^{\downarrow\downarrow} &= (E^*_\rho W + \cdots + E^*_{\rho+h} W) \cap (E_\tau W + \cdots + E_{\tau+d-h} W), \\
W_h^{\uparrow\downarrow} &= (E^*_{\rho+d-h} W + \cdots + E^*_{\rho+d} W) \cap (E_\tau W + \cdots + E_{\tau+d-h} W), \\
W_h^{\downarrow\uparrow} &= (E^*_\rho W + \cdots + E^*_{\rho+h} W) \cap (E_{\tau+h} W + \cdots + E_{\tau+d} W), \\
W_h^{\uparrow\uparrow} &= (E^*_{\rho+d-h} W + \cdots + E^*_{\rho+d} W) \cap (E_{\tau+h} W + \cdots + E_{\tau+d} W).
\end{aligned}$$

We remark that the sum (7) is not orthogonal in general.

Let W denote an irreducible T-module with diameter d. Then the following 1,2 hold for $0 \leq h, \ell \leq d$ such that $h + \ell \neq d$.

1. The subspaces $W_h^{\downarrow\downarrow}$ and $W_\ell^{\uparrow\uparrow}$ are orthogonal with respect to the standard Hermitian form.
2. The subspaces $W_h^{\downarrow\uparrow}$ and $W_\ell^{\uparrow\downarrow}$ are orthogonal with respect to the standard Hermitian form.

For $-1 \leq i, j \leq D$ we define

$$\begin{aligned}
V_{i,j}^{\downarrow\downarrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV), \\
V_{i,j}^{\uparrow\downarrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_0V + \cdots + E_jV), \\
V_{i,j}^{\downarrow\uparrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V), \\
V_{i,j}^{\uparrow\uparrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_DV + \cdots + E_{D-j}V).
\end{aligned}$$

In each of the above four equations we interpret the right-hand side to be 0 if $i = -1$ or $j = -1$.

For $\mu,\nu \in \{\downarrow,\uparrow\}$ and $0 \le i,j \le D$,

we have $V^{\mu\nu}_{i-1,j} \subseteq V^{\mu\nu}_{i,j}$ and $V^{\mu\nu}_{i,j-1} \subseteq V^{\mu\nu}_{i,j}$.

Therefore

$$V^{\mu\nu}_{i-1,j} + V^{\mu\nu}_{i,j-1} \subseteq V^{\mu\nu}_{i,j}.$$

We define $\tilde{V}^{\mu\nu}_{i,j}$ as

$$\tilde{V}^{\mu\nu}_{i,j} = (V^{\mu\nu}_{i-1,j} + V^{\mu\nu}_{i,j-1})^{\perp} \cap V^{\mu\nu}_{i,j}.$$

The following holds for $\mu, \nu \in \{\downarrow, \uparrow\}$:

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu\nu} \qquad \text{(direct sum)}. \tag{8}$$

Definition We call the sum (8) the (μ, ν)–**split decomposition** of V with respect to x.

We remark that the decomposition (8) is not orthogonal in general.

Let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then for $0 \leq h \leq d$ and $0 \leq i, j \leq D$ the following 1–4 hold.

1. $W_h^{\downarrow\downarrow} \subseteq \tilde{V}_{i,j}^{\downarrow\downarrow}$ if and only if $i = \rho + h$ and $j = \tau + d - h$.
2. $W_h^{\uparrow\downarrow} \subseteq \tilde{V}_{i,j}^{\uparrow\downarrow}$ if and only if $i = D - \rho - d + h$ and $j = \tau + d - h$.
3. $W_h^{\downarrow\uparrow} \subseteq \tilde{V}_{i,j}^{\downarrow\uparrow}$ if and only if $i = \rho + h$ and $j = D - \tau - h$.
4. $W_h^{\uparrow\uparrow} \subseteq \tilde{V}_{i,j}^{\uparrow\uparrow}$ if and only if $i = D - \rho - d + h$ and $j = D - \tau - h$.

Fix an orthogonal direct sum decomposition of the standard module V of Γ into irreducible T-modules:

$$V = \sum_{W} W. \qquad (9)$$

Then the following 1–4 hold for $0 \leq i, j \leq D$.

1. $\tilde{V}_{i,j}^{\downarrow\downarrow} = \sum W_h^{\downarrow\downarrow}$, where the sum is over all ordered pairs (W, h) such that W is assumed in (9) with endpoint $\rho \leq i$, dual endpoint $\tau = i + j - \rho - d$, diameter $d \geq i - \rho$, and $h = i - \rho$.

2. $\tilde{V}_{i,j}^{\uparrow\downarrow} = \sum W_h^{\uparrow\downarrow}$, where the sum is over all ordered pairs (W, h) such that W is assumed in (9) with endpoint $\rho \leq D - i$, dual endpoint $\tau = i + j + \rho - D$, diameter $d \geq D - \rho - i$, and $h = \rho + d - D + i$.
3. $\tilde{V}_{i,j}^{\downarrow\uparrow} = \sum W_h^{\downarrow\uparrow}$, where the sum is over all ordered pairs (W, h) such that W is assumed in (9) with endpoint $\rho \leq i$, dual endpoint $\tau = \rho + D - i - j$, diameter $d \geq i - \rho$, and $h = i - \rho$.
4. $\tilde{V}_{i,j}^{\uparrow\uparrow} = \sum W_h^{\uparrow\uparrow}$, where the sum is over all ordered pairs (W, h) such that W is assumed in (9) with endpoint $\rho \leq D - i$, dual endpoint $\tau = 2D - \rho - d - i - j$, diameter $d \geq D - \rho - i$, and $h = \rho + d - D + i$.

We show that with respect to the standard Hermitian form the $(\downarrow,\downarrow)$–split decomposition (resp. $(\downarrow,\uparrow)$–split decomposition) and the $(\uparrow,\uparrow)$–split decomposition (resp. $(\uparrow,\downarrow)$–split decomposition) are **dual** in the following sense.

Main Theorem The following 1,2 hold for $0 \leq i, j, r, s \leq D$.

1. $\tilde{V}^{\downarrow\downarrow}_{i,j}$ and $\tilde{V}^{\uparrow\uparrow}_{r,s}$ are orthogonal unless $i + r = D$ and $j + s = D$.
2. $\tilde{V}^{\downarrow\uparrow}_{i,j}$ and $\tilde{V}^{\uparrow\downarrow}_{r,s}$ are orthogonal unless $i + r = D$ and $j + s = D$.

Definition With reference to Notational convention, for $0 \leq i, j \leq D$ and for $\mu, \nu \in \{\downarrow, \uparrow\}$ we define $E^{\mu\nu}_{i,j} \in \mathrm{Mat}_X(\mathbb{C})$ so that

$$\begin{aligned} (E^{\mu\nu}_{i,j} - I)\tilde{V}^{\mu\nu}_{i,j} &= 0, \\ E^{\mu\nu}_{i,j}\tilde{V}^{\mu\nu}_{r,s} &= 0 \text{ if } (i,j) \neq (r,s) (0 \leq r, s \leq D). \end{aligned}$$

In other words $E^{\mu\nu}_{i,j}$ is the projection from V onto $\tilde{V}^{\mu\nu}_{i,j}$. We note that $E^{\mu\nu}_{i,j}V = \tilde{V}^{\mu\nu}_{i,j}$.

Lemma With reference to Notational convention, the following 1,2 hold for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j, r, s \leq D$.

1. $\sum_{i=0}^{D} \sum_{j=0}^{D} E^{\mu\nu}_{i,j} = I$.
2. $E^{\mu\nu}_{i,j} E^{\mu\nu}_{r,s} = \delta_{ir}\delta_{js} E^{\mu\nu}_{i,j}$.

Theorem With reference to Notational convention, the following 1-3 hold for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$.

1. $\overline{E^{\mu\nu}_{i,j}} = E^{\mu\nu}_{i,j}$.
2. $(E^{\downarrow\downarrow}_{i,j})^t = E^{\uparrow\uparrow}_{D-i,D-j}$.
3. $(E^{\downarrow\uparrow}_{i,j})^t = E^{\uparrow\downarrow}_{D-i,D-j}$.

Thank you for your attention!