# Recent Developments in Regular Maps

Shao-Fei Du
School of Mathematical Sciences
Capital Normal University
Beijing 100037, P. R. China

.

A 2-cell embedding of a graph into an orientable or nonorientable closed surface is called regular if its automorphism group acts regularly on its arcs and flags respectively. One of central problem in topological graph theory is to classify regular maps by given underlying graphs or automorphism groups. In this talk, we shall present some recent results in regular maps.

# 1 Groups acting on surfaces

## 2.1. Surfaces and Embeddings

*2-manifold M*: a topological space $M$ which is Hausdorf and is covered by countably many open sets isomorphic to either 2-dim open ball or 2-dim half-ball;

*Closed 2-manifold M:* compact, boundary is empty;

*Surface S:* closed, connected 2-manifold;

*Classification of Surfaces:*

(i) Orientable Surfaces: $S_g$, $g = 0, 1, 2, \cdots$ ,
$v + f - e = 2 - 2g$

(ii) Nonorientable Surfaces: $N_k$, $k = 0, 1, 2, \cdots$ ,
$v + f - e = 2 - k$

*Embeddings* of a graph $X$ in the surface is a continuous one-to-one function $i : X \rightarrow S$.

*2-cell Embeddings*: each region is homemorphic to an open disk.

The primitive objective of topological graph theory is to draw a graph on a surface so that no two edges cross.

**Topological Map $\mathcal{M}$:** a 2-cell embedding of a graph into a surface. The embedded graph $X$ is called the *underlying graph* of the map.

*Automorphism* of a map $\mathcal{M}$ : an automorphism of the underlying graph $X$ which can be extended to self-homeomorphism of the surface.

*Automorphism group* $\mathrm{Aut}\,(\mathcal{M})$ : all the automorphisms of the map $\mathcal{M}$.

Remark: $\mathrm{Aut}\,(\mathcal{M})$ acts semi-regularly on the arcs of $X$.

*Regular Map:* $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on the arcs of $X$.

Three main research directions:

1. Classifying regular maps by groups;

2. Classifying regular maps by underlying graphs

3. Classifying regular maps by genus

## 2.2 Combinatorial and Algebraic Map

## Combinatorial Orientable Map:

connected simple graph $\mathcal{G} = \mathcal{G}(V, D)$, with vertex set $V = V(\mathcal{G})$, dart (arc) set $D = D(\mathcal{G})$.

*arc-reversing involution* $L$: interchanging the two arcs underlying every given edge.

*rotation R:* cyclically permutes the arcs initiated at $v$ for each vertex $v \in V(\mathcal{G})$.

*Map $\mathcal{M}$ with underlying graph $\mathcal{G}$:*
the triple $\mathcal{M} = \mathcal{M}(\mathcal{G};\ R,\ L)$.

*Remarks: Monodromy group* $\mathrm{Mon}(\mathcal{M}) := \langle R, L \rangle$ acts transitively on $D$.

Given two maps

$$\mathcal{M}_1 = \mathcal{M}(\mathcal{G}_1;\ R_1,\ L_1),\ \mathcal{M}_2 = \mathcal{M}(\mathcal{G}_2;\ R_2\, L_2),$$

*Map isomorphism:* bijection $\phi : D(\mathcal{G}_1) \to D(\mathcal{G}_2)$ such that

$$L_1\phi = \phi L_2, R_1\phi = \phi R_2$$

*Automorphism* $\phi$ of $\mathcal{M}$ : if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

*Automorphism group*: $\mathrm{Aut}\,(\mathcal{M})$

Remarks: $\mathrm{Aut}\,(\mathcal{M}) = C_{S_D}(\mathrm{Mon}(\mathcal{M}))$;
$\mathrm{Aut}\,(\mathcal{M})$ acts semi-regularly on $D$,

*Regular Map:* $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on $D$.

Remarks: For regular map, we have

(i) $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$;

(ii) $\mathrm{Aut}\,(\mathcal{M})$ and $\mathrm{Mon}(\mathcal{M})$ on $D$ can be viewed as the right and the left regular representations of an abstract group $G = \mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$

**Algebraic Orientable Maps:**

*Coset graph:* Group $G$, $H \leq G$ free-core, $B = HgH$ with $B = B^{-1}$. Define *coset graph* $\mathcal{G} = \mathcal{G}(G; H, B)$ by
$V(\mathcal{G}) = \{Hg \bigm| g \in G\}$
$D(\mathcal{G}) = \{(Hg, Hbg) \bigm| b \in B, g \in G\}$.

**Definition:** *Let* $G = \langle r, \ell \rangle$ *be a finite two-generator group with* $\ell^2 = 1$ *and* $\langle r \rangle \cap \langle r \rangle^\ell = 1$. *By an algebraic map* $\mathcal{M}(G;\, r, \ell) = (\mathcal{G};\, R)$, *we mean the map whose underlying graph is the coset graph* $\mathcal{G} = \mathcal{G}(G; \langle r \rangle, \langle r \rangle \ell \langle r \rangle)$ *and rotation* $R$ *is determined by*

$$(\langle r \rangle g, \langle r \rangle l r^i g)^R = (\langle r \rangle g, \langle r \rangle l r^{i+1} g),$$

*for any* $g \in G$.

see

S.F. Du, J.H. Kwak and R. Nedela, A Classification of regular embeddings of graphs of order a product of two primes, *J. Algeb. Combin.* **19**(2004), 123–141.

(i) any algebraic map $\mathcal{M}$ is regular with $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M}) \cong G$.

(ii) Each regular map can be represented an algebraic map.

(iii) $\mathcal{M}(G; r_1, \ell_1) \cong \mathcal{M}(G;\ r_2, \ell_2)$ if and only if there exists an element $\sigma \in \mathrm{Aut}\,(G)$ such that $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

Classify all regular maps of a given underlying arc-transitive graph $\mathcal{G}$ with valency $s$ in the following two steps:

(1) Find the representatives $G$ (as abstract groups) of the isomorphic classes of arc-regular subgroups of $\mathrm{Aut}\,(\mathcal{G})$ with cyclic vertex-stabilizers.

(2) For each group $G$ given in (1), determine all the algebraic regular maps $\mathcal{M}(G; r, \ell)$ with underlying graphs isomorphic to $\mathcal{G}$, or equivalently, determine the representatives of the orbits of $\mathrm{Aut}\,(G)$ on the set of generating pairs $(r, \ell)$ of $G$ such that $|r| = n$, $|\ell| = 2$ and $\mathcal{G}(G; \langle r\rangle, \langle r\rangle\ell\langle r\rangle) \cong \mathcal{G}$.

**Combinatorial Nonorientable Map:**

**Definition 1.1** *For a given finite set $F$ and three fixed-point free involutory permutations $t, r, \ell$ on $F$, a quadruple $\mathcal{M} = \mathcal{M}(F; t, r, \ell)$ is called a combinatorial map if they satisfy two conditions: (1) $t\ell = \ell t$; (2) the group $\langle t, r, \ell \rangle$ acts transitively on $F$.*

$F$: *flag* set;

$t, r, \ell$ are called *longitudinal, rotary, and transverse involution,* respectively.

$\mathrm{Mon}(\mathcal{M}) = \langle t, r, \ell \rangle$: *Monodromy group* of $\mathcal{M}$,

*vertices, edges* and *face-boundaries* of $\mathcal{M}$ to be orbits of the subgroups $\langle t, r \rangle$, $\langle t, \ell \rangle$ and $\langle r, \ell \rangle$, respectively.

The incidence in $\mathcal{M}$ can be represented by nontrivial intersection.

The map $\mathcal{M}$ is called *unoriented*.

the even-word subgroup $\langle tr, r\ell \rangle$ of $\mathrm{Mon}(\mathcal{M})$ has the index at most 2.

*orientable*: if the index is 2,

*nonorientable*: if the index is 1

Given two maps $\mathcal{M}_1 = \mathcal{M}(F_1; t_1, r_1, \ell_1)$ and $\mathcal{M}_2 = \mathcal{M}_2(F_2; t_2, r_2, \ell_2)$,

*Map isomorphism:* bijection $\phi : F_1 \to F_2$ such that

$$\phi t_1 = t_2 \phi, \quad \phi r_1 = r_2 \phi, \quad \phi \ell_1 = \ell_2 \phi.$$

*Automorphism* of $\mathcal{M}$ : if $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$;

*Automorphism group*: $\mathrm{Aut}\,(\mathcal{M})$

Remarks: $\mathrm{Aut}\,(\mathcal{M}) = C_{S_F}(\mathrm{Mon}(\mathcal{M}))$;
$\mathrm{Aut}\,(\mathcal{M})$ acts semi-regularly on $F$,

*Regular Map:* $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on $F$.

Remarks: For regular map, we have
(i) $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}(\mathcal{M})$;
(ii) $\mathrm{Aut}\,(\mathcal{M})$ and $\mathrm{Mon}(\mathcal{M})$ on $F$ can be viewed as the right and the left regular representations of an abstract group $G$.
(iii) $\mathcal{M}(G; t_1, r_1, \ell_1) \cong \mathcal{M}(G; t_2, r_2, \ell_2)$ if and only if there exists $\sigma \in \mathrm{Aut}\,(G)$ such that $t_1^\sigma = t_2$, $r_1^\sigma = r_2$ and $\ell_1^\sigma = \ell_2$.

## 2.3. Classify regular maps by given graphs

1. Complete graphs:

Orientrable:

N.L. Biggs, Classification of complete maps on orientable surfaces, *Rend. Mat.* (6) **4** (1971), 132-138.
L.D. James and G.A. Jones, Regular orientable imbeddings of complete graphs, *J. Combin. Theory Ser. B* **39** (1985), 353–367.

The proof uses the characterization of sharply doubly transitive permutation groups, $n = p^k$ and $G = \mathrm{AGL}(1, p^k)$.

Nonorientrable:

S. E. Wilson, Cantankerous maps and rotary embeddings of $K_n$, *J. Combin. Theory Ser. B* **47** (1989), 262–273.

$n$ must be of order 3, 4 or 6.

(2) Complete multipartite graphs $K_n[\bar{K}_p]$, $p$ prime,

Orientable maps.

S.F. Du, J.H. Kwak and R. Nedela, Regular embeddings of complete multipartite graphs, *European J. Combin.* **26** (2005), 505–519.

Independent of CFSG.

(3) Graphs of order $pq$

Orienable maps:

S.F. Du, J.H. Kwak and R. Nedela, A Classification of regular embeddings of graphs of order a product of two primes, *J. Algeb. Combin.* **19**(2004), 123–141.

Independent of CFSG.

Nonorienable maps:

S.F. Du and F.R.Wang, Nonorientable regular embeddings of graphs of order a product of two distinct primes, submitted to *J. Graph Theory.*

depends on CFSG.

(4) Complete bipartite graphs $K_{n,n}$:

Nonorientable Maps:

J.H.Kwak and Y.S.Kwon, Classification of nonorientable regular embeddings complete bipartite graphs, Submitted.

Orientable Maps

Regular embeddings of $K_{n,n}$ are very important, which have been studied in connection with various branches of mathematics including Riemann surfaces and algebraic curves, Galois groups, see survey paper:

G.A. Jones, Maps on surfaces and Galois groups, *Math. Slovaca* **47** (1997), 1-33.

Classification processes was begun by Nedela, Škovuera and Zlatoš:

R. Nedela, M. Škoviera and A. Zlatoš, Regular embeddings of complete bipartite graphs, *Discrete Math.*, **258** (1-3), 2002, p. 379–381.

*$n$ is a product of two primes:*

J.H. Kwak and Y.S. Kwon, Regular orientable embeddings of complete bipartite graphs, *J. Graph Theory* **50** (2005), 105–122.

*Reflexible maps:*

J. H. Kwak and Y. S. Kwon, Classification of reflexible regular embeddings and self-Petrie dual regular embeddings of complete bipartite graphs, Submitted.

$(n, \phi(n)) = 1$*:*

G.A. Jones, R. Nedela and M. Škoviera, Complete bipartite graphs with unique regular embeddings, submitted.

$n = p^k$*, $p$ is odd prime:*

G.A. Jones, R. Nedela and M. Škoviera, Regular Embeddings of $K_{n,n}$ where $n$ is an odd prime power, *European J. Combin.* **28** (2007), 1863-1875.

$n = 2^k$,

S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. I: Metacyclic case, *European J. Combin.* **28** (2007), 1595-1608.

S.F. Du, G.A.Jones, J.H. Kwak, R. Nedela and M. Škoviera, Regular embeddings of $K_{n,n}$ where n is a power of 2. II: Nonmetacyclic case, submitted

*Any n*:

G.A. Jones, Regular embeddings of complete bipartite graphs: classification and enumeration, (preparation, 2007.)

The key point for this work is to determine the structure of group

$G = \langle a \rangle \langle b \rangle$, where $|a| = |b| = n$, $\langle a \rangle \cap \langle b \rangle = 1$, and $a^\alpha = b$ for some involution $\alpha$ in $\mathrm{Aut}\,(G)$.

If $n = p^k$ and $p$ is odd, then a result of Huppert implies that such a group $G$ must be metacyclic.

If $n = 2^k$, we need to classify non-metacyclic case.

**Theorem 1.2** *(Du, Jones, Kwak, Nedela, Skoviera)*

*Suppose that $G = \langle a \rangle \langle b \rangle$, where $|a| = |b| = 2^e$, $e \geq 2$, $\langle a \rangle \cap \langle b \rangle = 1$, and $a^\alpha = b$ for some involution $\alpha$ in* $\mathrm{Aut}\,(G)$. *Then one of the following cases hold:*

(1) *$G$ is metacyclic and $G$ has presentation*

$$G_1(e, f) = \langle h, g \mid h^{2^e} = g^{2^e} = 1, h^g = h^{1+2^f} \rangle$$

*where $f = 2, \ldots, e$, and we may set $a = g^m$ and $b = g^m h$, where $m$ is odd, $1 \leq m \leq 2^{e-f}$;*

(2) *$G$ is not metacyclic, $G' \cong C_2$, and $G$ has presentation*

$$G_2 = \langle a, b \mid a^4 = b^4 = [a^2, b] = [b^2, a] = 1, [b, a] = a^2 b^2 \rangle;$$

(3) *$G'$ is generated by two elements, and $G$ has presentation*

$$G_3(e, k, l) = \langle a, b \mid a^{2^e} = b^{2^e} = [b^2, a^2] = 1,$$
$$[b, a] = a^{2+k2^{e-1}} b^{-2+k2^{e-1}},$$
$$(b^2)^a = a^{l2^{e-1}} b^{-2+l2^{e-1}}, (a^2)^b = a^{-2+l2^{e-1}} b^{l2^{e-1}} \rangle,$$

*where* $e \geq 3$*, and* $k, l \in \{0, 1\}$*. Moreover,* $G_3(e, 0, 1) \cong G_3(e, 1, 1)$*.*

Generally, for the structure of groups which is a product of two abelian groups, here I recommend the following papers:

B. Huppert, Über das Produkt von paarweise vertauschbaren zyklischen Gruppen, *Math. Z.* **58** (1953), 243–264.

N. Itô, Über das Produkt von zwei abelschen Gruppen, *Math. Z.* **62** (1955), 400–401.

N. Itô, Über das Produkt von zwei zyklischen 2-Gruppen, *Publ. Math. Debrecen* **4** (1956), 517–520.

M. D. E. Conder and I. M. Isaacs, Derived subgroups of product an abelian and a cyclic subgroup, *J. London Math. Soc.* **69** (2004), 333–348.

(4) *n-dimensional hypercubes* $Q_n$*:*

Graph $Q_n$ : vertex set $V = V(n, 2)$, while two vertices $\mathbf{x_1}$ and $\mathbf{x_2}$ are adjacent if and only if $\mathbf{x_1} + \mathbf{x_2}$ is an unit vector. This graph has valency $n$ and automorphism group $\mathrm{Aut}\,(Q_n) = Z_2^n : S_n$

*Nonorientable maps:*

Y.S. Kwon and R. Nedela, Non-existence of nonorientable regular embedings of $n-$dimensional cubes, to appear in *Discrete Math.*.

Only $Q_2$, which an embedding in projective plane.

*Orientable maps:*

*$n$ is odd:*

S.F. Du, J.H. Kwak and R. Nedela, Classification of regular embeddings of hypercubes of odd dimension, *Discrete Math.* **307**(1) (2007), 119-124.

*$n = 2m$, $m$ is odd:* Jing Xu, Classification of regular embeddings of hypercubes of dimension $2m$, when $m$ is odd, *Science in China, 2007*

**Problem 1.3** *Classify regular embeddings of hypercubes dimension $n$ for $n = 2^k m$,$k \geq 2$ and $m \geq 3$ is odd.*

Key point is to determine the arc regular subgroups $\langle r, \ell\rangle$ of $\mathrm{Aut}\,(Q_n) = Z_2^n : S_n$ s.t. $|r| = n$ and $|\ell| = 2$.

## 2.3. Classify regular maps by given groups

**Question 1.4** *Study a finite group $G$, as quotients of triangle groups, realize these groups as automorphism groups of compact Riemann surfaces.*

G.A. Jones and D. Singerman, *Complex function: an algebraic and geometric viewpoint,* Cambridge Univ. Press, 1987.

**Question 1.5** *Given a group $G$, classify all the regular maps with the automorphism groups isomorphic to $G$.*

1. Orientable cases

(i) $G = \mathrm{PSL}(2, q)$:

Macbeath described all triples $x, y, z$ with $xyz = 1$ generating $\mathrm{PSL}(2, q)$ (in fact $SL(2, q)$) in terms of their orders.

A.M. Macbeath, Generators of the linear fractional groups. 1969 Number Theory Proc. Sympos. Pure Math., Vol. XII, Houston, Tex., 1967, 14–32 Amer. Math. Soc., Providence, R.I. 20.75 (30.00)

Other results:

A.A. Albert and J. Thompson, Two-element generation of the projective unimodular group, *Illinois J. Math.* **3**(1959), 421–439.

M. Downs, Some enumerations of regular hypermaps with automorphism group isomorphic to $\mathrm{PSL}_2(q)$, *Quart. J. Math. Oxford Ser.* (2)**48**(1997), 39–58.

D. Garbe, Über eine Klasse von arithmetisch definierbaren Normalteilern der Modulgruppe. (German) *Math. Ann.* (3)**235**(1978), 195–215.

H. Glover and D. Sjerve, The genus of $\mathrm{PSL}_2(q)$, *J. Reine Angew. Math.* **380**(1987), 59–86.

H. Glover and D. Sjerve, Representing $\mathrm{PSL}_2(p)$ on a Riemann surface of least genus, *Enseign. Math.* (2)**31** (1985), 305–325.

U. Langer and R. Rosenberger, Erzeugende endlicher

projektiver linearer Gruppen (German), *Results Math.* **15**(1989), 119–148.

C.H. Sah, Groups realted to compact Riemann surfaces, *Acta Math.* **1969**, 13–42.

D.B. Surowski, Vertex-transitive triangulations of compact orientable 2-manifolds, *J. Combin. Theory Ser. B* (3)**39** (1985), 371–375.

(ii) Hurwitz groups

(2, 3, 7) triangle group, where many cases of finite (usually simple) groups have been recently shown to be quotients or non-quotients, Hurwitz groups or non-Hurwitz groups

$A_n$ and $S_n$:

M.D.E. Conder, The summetric genus of alternating and symmetric groups, *J. Combin. Theroy Ser. B* **39**(1985), 179-186.

Suzuki groups:

G.A. Jones and S.A. Silver, Suzuki groups and surfaces, *J. London Math. Soc.* (2)**48**(1993), 117–125.

Ree groups:

G.A. Jones, Ree groups and Riemann surfaces, *J. Algebra*, (1)**165**(1994), 41–62.

C.H. Sah, Groups realted to compact Riemann surfaces, *Acta Math.* **1969**, 13–42.

Survey paper:

M.D.E. Conder, Hurwitz groups: A brief survey, *Bull. Amer. Math. Soc.* **23**(1990), 359-370.

2. Nonorientalbe cases

very few results.

Singerman showed that PSL$(2, q)$ is a homomorphic image of the extended modular group for all $q$ except for $q = 7$, 11 and $3^n$, where $n = 2$ or $n$ is odd and he gave some applications to group actions on surfaces.

D. Singerman, PSL$(2, q)$ as an image of the extended modular group with applications to group actions on surfaces, Groups—St. Andrews 1985. *Proc. Edinburgh Math. Soc.* (2)**30**(1987), no. 1, 143–151.

S.F. Du and J.H.Kwak, Groups PSL$(3,p)$ and Nonorientable Regular Maps, *Journal of Algebra.* in reversion, 2007, 23 pages.

**Theorem 1.6** *For a prime $p$, set $G = \mathrm{SL}(3,p)$ and $\overline{G} = \mathrm{PSL}(3,p)$. Let $\mathcal{M}$ be a nonorientable regular map with the automorphism group isomorphic to $\overline{G}$. Then $\mathcal{M}$ is isomorphic to one of the maps $\mathcal{M}(\alpha,\beta) = \mathcal{M}(\overline{G};\bar{t},\overline{r},\bar{l}_{(\alpha,\beta)})$, where*

$$t = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\ell_{(\alpha,\beta)} = \begin{pmatrix} \alpha & \beta & 0 \\ \beta^{-1}(1-\alpha^2) & -\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

*where $p \geq 5$, $\alpha, \beta \in F_p^*$ and $\alpha \neq \pm 1$. Moreover, $\mathcal{M}(\alpha_1,\beta_1) \cong \mathcal{M}(\alpha_2,\beta_2)$ if and only if $(\alpha_1,\beta_1) = (\alpha_2,\beta_2)$. In particular, there are $p^2-4p+3$ maps, each of which has the simple underlying graph of valency $p$.*

**Problem 1.7** *Given a finite simple group $G$, classify all the regular maps with the automorphism groups isomorphic to $G$.*

**Thank you very much !**