Equilibrium : Jacobian Matrix
– ODE of the Dynamical System and its stability

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An **equilibrium** (or **equilibrium point**) of a dynamical system generated by an autonomous system of ordinary differential equations (ODEs) is a solution that does not change with time. Geometrically, **equilibria** are points in the system's phase space. Equilibria are sometimes called **fixed points or steady states**. Most mathematicians refer to equilibria as time-independent solutions of ODEs, and to fixed points as time-independent solutions of iterated maps \((t+1)=f(x(t))\).

In the economic system, price don’t move anywhere at the equilibrium point.

For example, each motionless pendulum position corresponds to an equilibrium of the corresponding equations of motion, one is stable, the other one is not.
Supply and demand is an economic model based on price, utility and quantity in a market. It concludes that in a competitive market, price will function to equalize the quantity demanded by consumers, and the quantity supplied by producers, resulting in an economic equilibrium of price and quantity.

The price P of a product is determined by a balance between production at each price (supply S) and the desires of those with purchasing power at each price (demand D). Along with a consequent increase in price (P) and quantity sold (Q) of the product.
Equilibrium: Simple Example (Supply-Demand Model)

How we can find equilibrium points from the supply-demand model?

Demand: \( f(x) = \frac{2}{x + 1} \)

Supply: \( g(x) = 0.5x^2 + 0.4 \)
Equilibrium: Simple Example (Supply-Demand Model)

The effects on price and quantity if supply and demand changes and the other remains constant or if both the demand and supply curves shift are as follow:

Supply/ Demand
Increase/Increase P↑Q↑
Increase/Decrease P↑Q↓
Increase/Constant P↑Q↑
Decrease/Increase P↑Q↑
Decrease/Decrease P↑Q↓
Decrease/Constant P↑Q↑
Constant/Increase P↑Q↓
Constant/Decrease P↑Q↓
Constant/Constant NC
? = Indeterminant.

We separate 4 phases in moving to find a equilibrium point.
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\( = \frac{2}{A3 + 1} \)

Supply: \( g(x) = 0.5x^2 + 0.4 \)
Equilibrium: Simple Example (Supply-Demand Model)

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**Equilibrium: Simple Example (Supply-Demand Model)**

Fail scenario: Don’t approach the equilibrium point.

In this case, first supply was determined by its prices and demands in the market and we must consider its effect in the real-market.
Equilibrium: Simple Example (Harvest Model)

In logistic growth model, we add the harvesting process from the growth objects.
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\[ \frac{dS}{dt} = \alpha(1 - \theta)S - H \]

\[ \theta = \frac{S}{\text{capacity}} \]

\[ = (1 - \frac{B5}{B2}) \times B1 = E1 \]
**Equilibrium: Simple Example (Harvest Model)**

Equilibrium point: To make a balance between the growth and the harvest of the objects.

Using the cobweb, we can find an equilibrium point.

Identity Function $G_1$: $g_1(x) = x$

Farm Function $G_2$: $g_2(x) = (1 + r)x - \frac{r}{L}x^2 - b$
**Equilibrium: Relationship with Jacobian Matrices**

The stability of typical equilibria of smooth ODEs is determined by the sign of real part of eigenvalues of the Jacobian matrix. These eigenvalues are often referred to as the 'eigenvalues of the equilibrium'. The Jacobian Matrix of a system of smooth ODEs is the matrix of the partial derivatives of the right-hand side with respect to state variables

\[ J = D_x f = f_x = \left( \frac{\partial f_i}{\partial x_j} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \]

where all derivatives are evaluated at the equilibrium point. Its eigenvalues determine linear stability properties of the equilibrium.

An equilibrium is asymptotically stable if all eigenvalues have negative real parts; it is unstable if at least one eigenvalue has positive real part.
In mathematics, in the study of dynamical systems, the **Hartman-Grobman theorem** or linearization theorem is an important theorem about the local behavior of dynamical systems in the neighborhood of a hyperbolic fixed point.

Basically the theorem states that the behavior of a dynamical system near a hyperbolic fixed point is qualitatively the same as the behavior of its linearization near the origin. Therefore when dealing with such fixed points we can use the simpler linearization of the system to analyze its behavior.

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth map with a hyperbolic fixed point \( p \).

Let \( A \) denote the linearization of \( f \) at point \( p \).

Then there exist a neighborhood \( U \) of \( p \) and a homeomorphism \( h : U \to \mathbb{R}^n \) such that

\[
  f |_U = h^{-1}Ah
\]

that is in a neighborhood \( U \) of \( p \), \( f \) is topologically conjugate to its linearization.

In general, even for infinitely differentiable maps \( f \), the homomorphism \( h \) need not to be smooth, nor even locally Lipschitz. However, it turns out to be Hölder continuous, with an exponent depending on the constant of hyperbolicity of \( A \).

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Hyperbolic Equilibrium: Hartman-Grobman Theorem for differential equations

Local phase portrait of a hyperbolic equilibrium of a non-linear system is equivalent to that of its linearization. This statement has a mathematically precise form known as the Hartman-Grobman Theorem. It says that solutions of

\[ x' = f(x) \]

in a small neighborhood of a hyperbolic equilibrium can be mapped with a homeomorphism (i.e., continuous map with a continuous inverse) onto solutions of the linear system

\[ y' = Jy \]

where \( J \) is the Jacobian matrix at the equilibrium. One says that these systems are locally topologically conjugate (equivalent). That is, adding nonlinear terms to a linear system at a hyperbolic equilibrium may distort but does not change qualitatively the phase portrait near the equilibrium.

If at least one eigenvalue of the Jacobian matrix is zero or has zero real part, then the equilibrium is said to be non-hyperbolic. **Non-hyperbolic equilibria are not robust (i.e., the system is not structurally stable):** Small perturbations can result in a local bifurcation of a non-hyperbolic equilibrium, i.e., it can change stability, disappear, or split into many equilibria. Some refer to such an equilibrium by the name of the bifurcation, e.g., saddle-node equilibrium.

In practice, one often has to consider non-hyperbolic equilibria with all eigenvalues having negative or zero real parts. These equilibria are sometimes referred to as being critical. Their stability cannot be determined from the signs of the eigenvalues of the Jacobian matrix; it depends on the nonlinear terms of \( f \).
Types of Equilibrium: One-Dimensional Space

Consider a one-dimensional (scalar) dynamical system

\[ x' = f(x), \quad x \in \mathbb{R}^1 \]

\[ J = f'(x) \]

Its equilibria are the zeros of the function \( f(x) \). An equilibrium is asymptotically stable when \( f(x) < 0 \), that is, the slope of \( f \) is negative. It is unstable when \( f(x) > 0 \). The left two equilibria in the figure are hyperbolic \((f'(x)=0)\), the others are non-hyperbolic because the slope (eigenvalue) is zero. Nevertheless, a non-hyperbolic equilibrium of a one-dimensional system is stable if the function changes the sign from positive to negative at the equilibrium.
Consider a two-dimensional (planar) system with smooth right-hand side

\[ x_1' = f_1(x_1, x_2) \]
\[ x_2' = f_2(x_1, x_2) \]

The Jacobian matrix has the form

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix}
\]

It has two eigenvalues, which are either both real or complex-conjugate. A hyperbolic equilibrium can be a

- **Node** when both **eigenvalues are real and of the same sign**. The node is stable when the eigenvalues are negative and unstable when they are positive. For the stable node, the eigenvalue(s) with minimal absolute value of the real part is called principle or leading; when the eigenvalues are different, all orbits but two tend to the node along the leading eigenvector (the picture is reversed for the unstable node);
- **Saddle** when **eigenvalues are real and of opposite signs**. The saddle is always unstable;
- **Focus** (sometimes called spiral point) when **eigenvalues are complex-conjugate**. The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.
The Jacobian matrix of a three-dimensional system has 3 eigenvalues, one of which must be real and the other two can be either both real or complex-conjugate. A hyperbolic equilibrium can be:

- **Node** when all eigenvalues are real and have the same sign; The node is stable (unstable) when the eigenvalues are negative (positive);
- **Saddle** when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable;
- **Focus-Node** when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive);
- **Saddle-Focus** when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type of equilibrium is always unstable.
There are many more types of non-hyperbolic equilibria, i.e., those that have at least one eigenvalue with zero real part, since the phase portrait in a small neighborhood of such equilibria also depends on the nonlinear terms of $f(x)$. Most of these equilibria do not have names or are named after the type of the bifurcation in which they play a role.

The **center equilibrium** occurs when a system has only two eigenvalues on the imaginary axis, namely, one pair of pure-imaginary eigenvalues. Centers in linear systems have families of concentric periodic orbits around them, as in the figure. Many refer to centers only in the context of two-dimensional systems or Hamiltonian systems. If all other eigenvalues have negative real parts, centers are neutrally stable but not asymptotically stable. A pair of pure-imaginary eigenvalues also occurs in Andronov-Hopf bifurcation, however due to the nonlinear terms, the neighborhood of such an equilibrium looks like a focus; it could be asymptotically stable (supercritical Andronov-Hopf bifurcation) or unstable (subcritical Andronov-Hopf bifurcation) even if the other eigenvalues have negative real parts.

The **saddle-node equilibrium** occurs in nonlinear systems with one zero eigenvalue when the system undergoes the saddle-node bifurcation, where a saddle and a node approach each other, coalesce into a single equilibrium (depicted in the figure), and then disappear. Saddle-nodes are always unstable.

The **Bogdanov-Takens equilibrium** occurs in nonlinear systems with 2 zero eigenvalues, typically when the system undergoes the Bogdanov-Takens bifurcation. It is also an unstable equilibrium.

**Examples of non-hyperbolic equilibria in $\mathbb{R}^2$**
References
