Unique Origin
Unique Future

Mathematical Modeling Lecture

Equilibrium : Jacobian Matrix
– ODE of the Dynamical System and its stability

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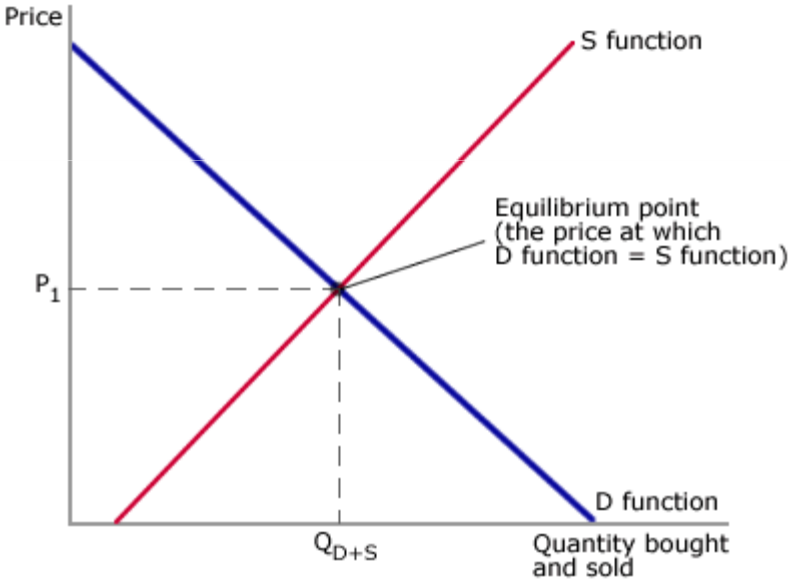
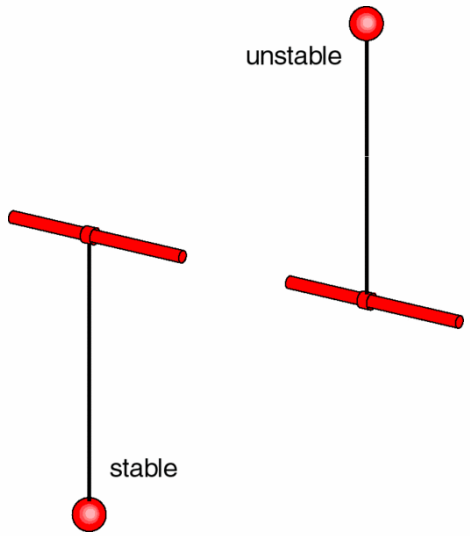
Equilibrium

An **equilibrium** (or **equilibrium point**) of a dynamical system generated by an autonomous system of ordinary differential equations (ODEs) is a solution that does not change with time. Geometrically, **equilibria** are points in the system's phase space. Equilibria are sometimes called **fixed points or steady states**. Most mathematicians refer to equilibria as time-independent solutions of ODEs, and to fixed points as time-independent solutions of iterated maps (t+1)=f(x(t)).



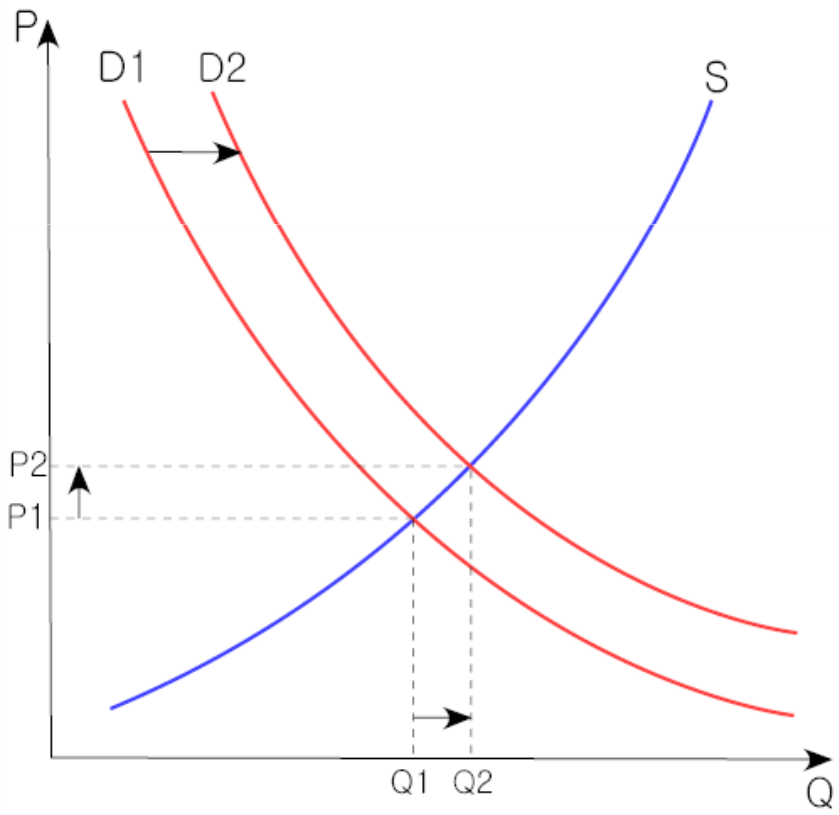
In the economic system, price don't move anywhere at the equilibrium point.



For example, each motionless pendulum position corresponds to an equilibrium of the corresponding equations of motion, one is stable, the other one is not.

Equilibrium: Simple Example (Supply-Demand Model)

Supply and demand is an economic model based on price, utility and quantity in a market. It concludes that in a competitive market, price will function to equalize the quantity demanded by consumers, and the quantity supplied by producers, resulting in an economic equilibrium of price and quantity.



The price P of a product is determined by a balance between production at each price (supply S) and the desires of those with purchasing power at each price (demand D). 2, along with a consequent increase in price (P) and quantity sold (Q) of the product.

Equilibrium: Simple Example (Supply-Demand Model)

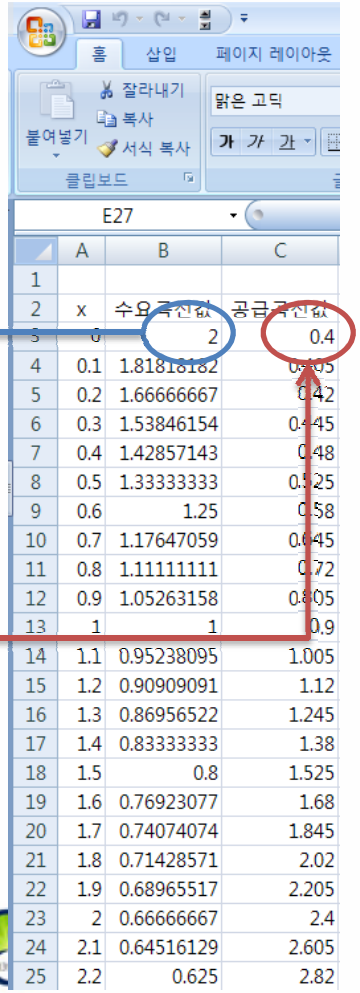
How we can find equilibrium points from the supply-demand model?

Demand : $f(x) = \dfrac{2}{x+1}$

=2/(A3+1)

=0.5*A3^2+0.4

Supply : $g(x) = 0.5x^2 + 0.4$



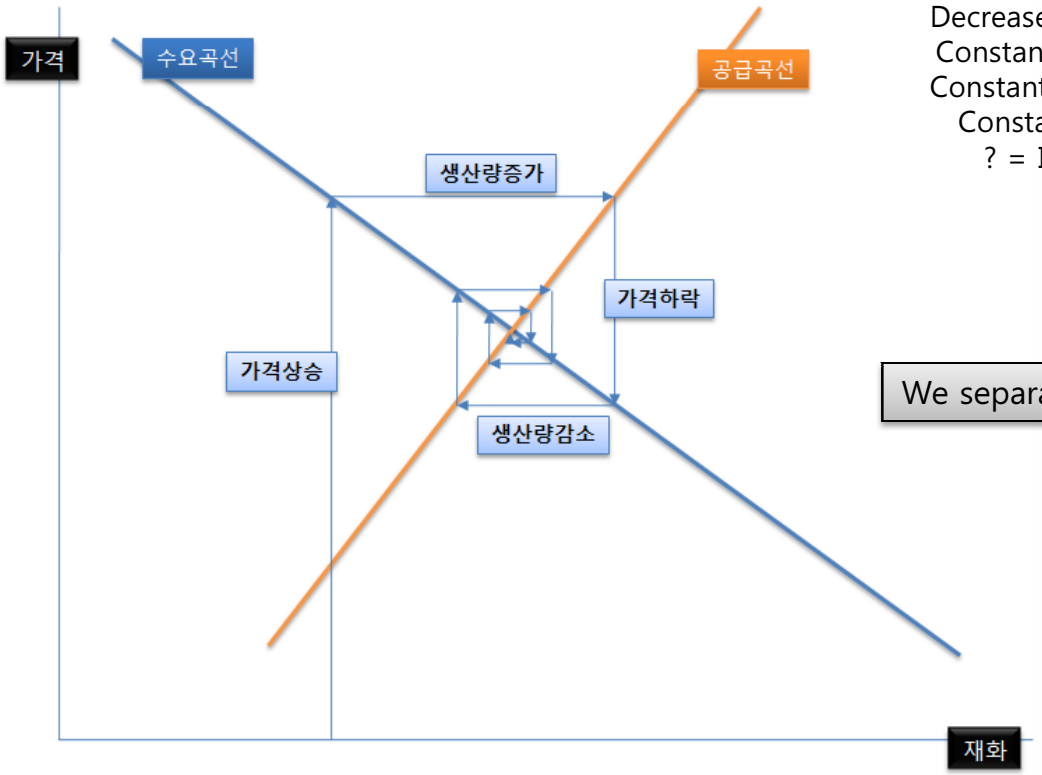
E27

	A	B	C
1			
2	x	수요곡선값	공급곡선값
3	0	2	0.4
4	0.1	1.81818182	0.405
5	0.2	1.66666667	0.42
6	0.3	1.53846154	0.445
7	0.4	1.42857143	0.48
8	0.5	1.33333333	0.525
9	0.6	1.25	0.58
10	0.7	1.17647059	0.645
11	0.8	1.11111111	0.72
12	0.9	1.05263158	0.805
13	1	1	0.9
14	1.1	0.95238095	1.005
15	1.2	0.90909091	1.12
16	1.3	0.86956522	1.245
17	1.4	0.83333333	1.38
18	1.5	0.8	1.525
19	1.6	0.76923077	1.68
20	1.7	0.74074074	1.845
21	1.8	0.71428571	2.02
22	1.9	0.68965517	2.205
23	2	0.66666667	2.4
24	2.1	0.64516129	2.605
25	2.2	0.625	2.82

Equilibrium: Simple Example (Supply-Demand Model)

The effects on price and quantity if supply and demand changes and the other remains constant or if both the demand and supply curves shift are as follow:

Supply/ Demand
Increase/Increase P?Q↑
Increase/Decrease P↑Q?
Increase/Constant P↓Q↑
Decrease/Increase P↑Q?
Decrease/Decrease P?Q↓
Decrease/Constant P↑Q↑
Constant/Increase P↓Q↓
Constant/Decrease P↑Q↓
Constant/Constant NC
? = Indeterminant.



We separate 4 phases in moving to find a equilibrium point.

Equilibrium: Simple Example (Supply-Demand Model)

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Supply : $g(x) = 0.5x^2 + 0.4$

Phases

Cobweb points

	A	B	C	D	E	F
1						
2	x	수요곡선값	공급곡선값	꺾임	x	y
3	0	2	0.4	0	0.36	0
4	0.1	1.81818182	0.405	1	0.36	1.470588
5	0.2	1.66666667	0.42	2	1.463276	1.470588
6	0.3	1.53846154	0.445	3	1.463276	0.811927
7	0.4	1.42857143	0.48	0	1.463276	0.811927
8	0.5	1.33333333	0.525	1	1.463276	0.811927
9	0.6	1.25	0.58	2	0.907664	0.811927
10	0.7	1.17647059	0.645	3	0.907664	1.048403
11	0.8	1.11111111	0.72	0	0.907664	1.048403
12	0.9	1.05263158	0.805	1	0.907664	1.048403
13	1	1	0.9	2	1.138774	1.048403
14	1.1	0.95238095	1.005	3	1.138774	0.935115
15	1.2	0.90909091	1.12	0	1.138774	0.935115
16	1.3	0.86956522	1.245	1	1.138774	0.935115
17	1.4	0.83333333	1.38	2	1.03452	0.935115
18	1.5	0.8	1.525	3	1.03452	0.983033
19	1.6	0.76923077	1.68	0	1.03452	0.983033
20	1.7	0.74074074	1.845	1	1.03452	0.983033
21	1.8	0.71428571	2.02	2	1.079845	0.983033
22	1.9	0.68965517	2.205	3	1.079845	0.96161
23	2	0.66666667	2.4	0	1.079845	0.96161
24	2.1	0.64516129	2.605	1	1.079845	0.96161
25	2.2	0.625	2.82	2	1.059821	0.96161

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Phases

Cobweb points

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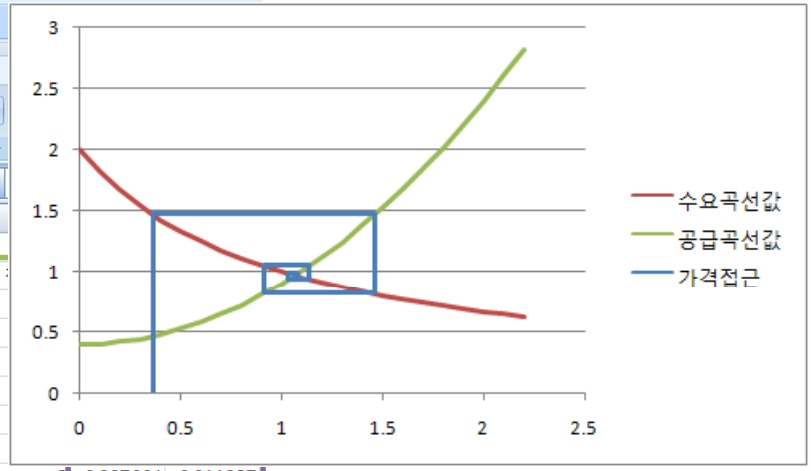

	A	B	C			
1						
2	x	수요곡선값	공급곡선값			
3	0	2	0.4			
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6	0.3	1.53846154	0.445			
7	0.4	1.42857143	0.48			
8	0.5	1.33333333	0.525			
9	0.6	1.25	0.58	2	0.907664	0.811927
10	0.7	1.17647059	0.645	3	0.907664	1.048403
11	0.8	1.11111111	0.72	0	0.907664	1.048403
12	0.9	1.05263158	0.805	1	0.907664	1.048403
13	1	1	0.9	2	1.138774	1.048403
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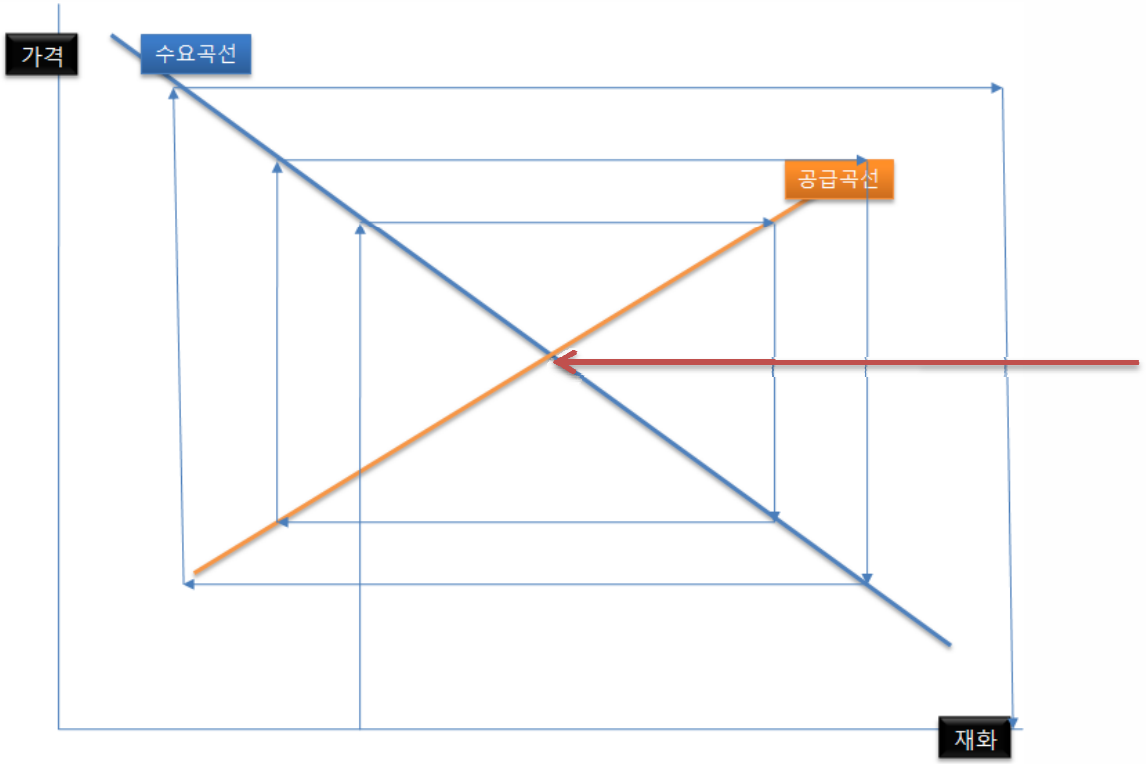
15	1.2	0.90909091	1.12	0	1.138774	0.935115
16	1.3	0.86956522	1.245	1	1.138774	0.935115
17	1.4	0.83333333	1.38	2	1.03452	0.935115
18	1.5	0.8	1.525	3	1.03452	0.983033
19	1.6	0.76923077	1.68	0	1.03452	0.983033
20	1.7	0.74074074	1.845	1	1.03452	0.983033
21	1.8	0.71428571	2.02	2	1.079845	0.983033
22	1.9	0.68965517	2.205	3	1.079845	0.96161
23	2	0.66666667	2.4	0	1.079845	0.96161
24	2.1	0.64516129	2.605	1	1.079845	0.96161
25	2.2	0.625	2.82	2	1.059821	0.96161

Equilibrium: Simple Example (Supply-Demand Model)

Fail scenario: Don't approach the equilibrium point.

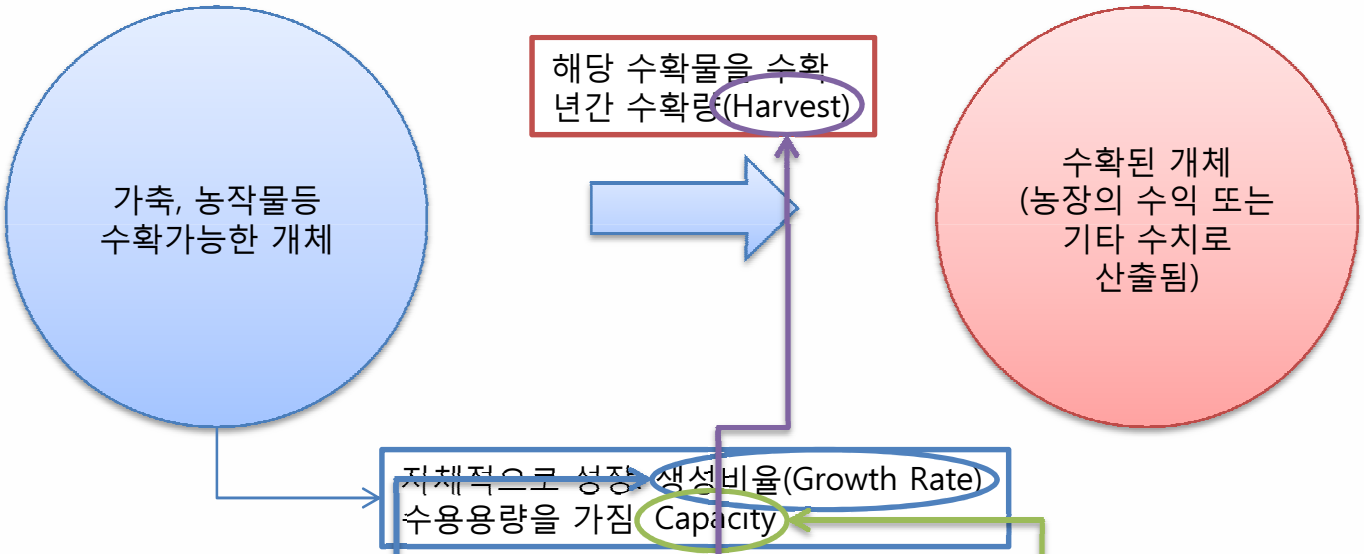
In this case, first supply was determined by its prices and demands in the market and we must consider its effect in the real-market.



Fail to approach the equilibrium point.

Equilibrium: Simple Example (Harvest Model)

In logistic growth model, we add the harvesting process from the growth objects.



수확가능한
개체수의 변화량 : $\frac{dS}{dt} = \alpha(1-\theta)S - H$ $\theta = \frac{S}{capacity}$

Equilibrium: Simple Example (Harvest Model)

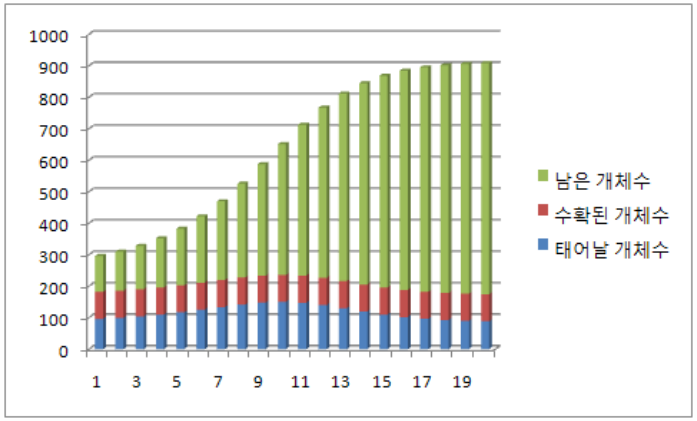
In logistic growth model, we add the harvesting process from the growth objects.

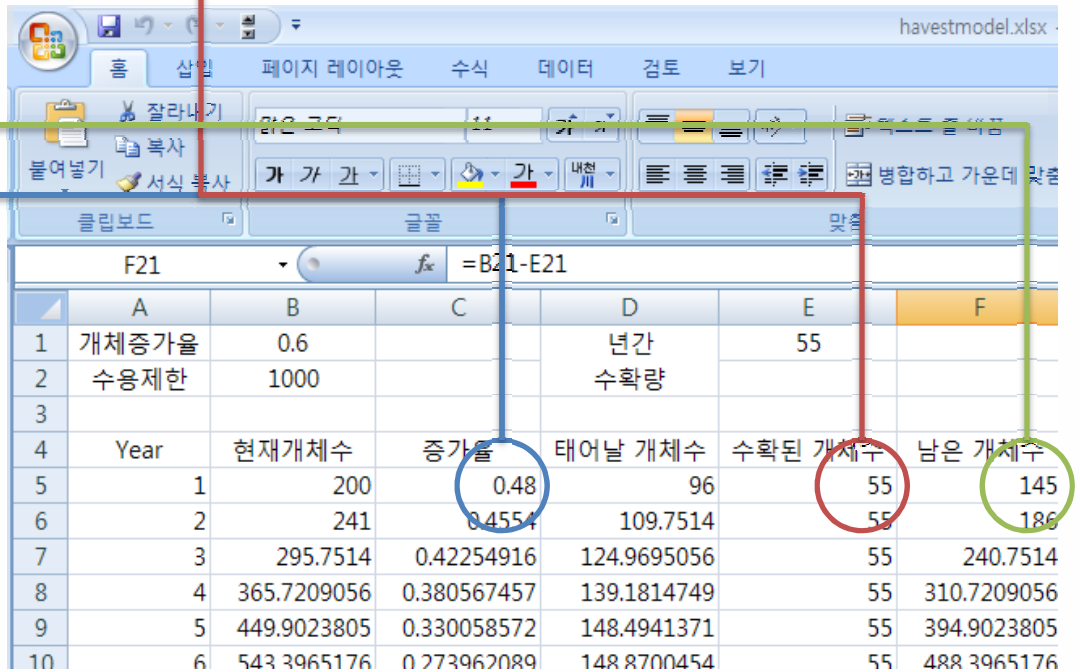
수확가능한
개체수의 변화량 :

$$\frac{dS}{dt} = \alpha(1-\theta)S - H \qquad \theta = \frac{S}{capacity}$$

=(1-B5/$B$2)*$B$1

=$E$1





	A	B	C	D	E	F
1	개체증가율	0.6		년간	55	
2	수용제한	1000		수확량		
3						
4	Year	현재개체수	증가율	태어날 개체수	수확된 개체수	남은 개체수
5	1	200	0.48	96	55	145
6	2	241	0.4554	109.7514	55	186
7	3	295.7514	0.42254916	124.9695056	55	240.7514
8	4	365.7209056	0.380567457	139.1814749	55	310.7209056
9	5	449.9023805	0.330058572	148.4941371	55	394.9023805
10	6	543.3965176	0.273962089	148.8700454	55	488.3965176

Equilibrium: Simple Example (Harvest Model)

Equilibrium point: To make a balance between the growth and the harvest of the objects.

Using the cobweb, we can find an equilibrium point.

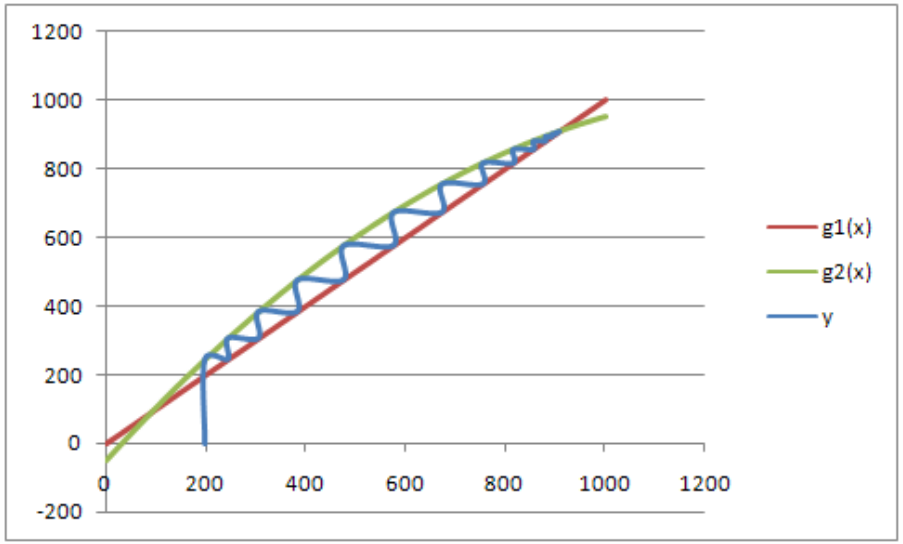
Identity Function G1 : $g_1(x) = x$

Farm Function G2 : $g_2(x) = (1+r)x - \frac{r}{L}x^2 - b$

개체 증가율

수용한계

수확량



개체증가율	0.8		년간	36		
수용제한	1000		수확량			
x	g1(x)	g2(x)		꺾어짐판정	x	y
0	0	-36		0	200	0
10	10	-18.08		1	200	292
20	20	-0.32		0	292	292
30	30	17.28		1	292	421.3888
40	40	34.72		0	421.3888	421.3888
50	50	52		1	421.3888	580.445023
60	60	69.12		0	580.445023	580.445023
70	70	86.08		1	580.445023	739.267902
80	80	102.88		0	739.267902	739.267902
90	90	119.52		1	739.267902	857.468599
100	100	136		0	857.468599	857.468599
110	110	152.32		1	857.468599	919.24156
120	120	168.48		0	919.24156	919.24156

Equilibrium: Relationship with Jacobian Matrices

The stability of typical equilibria of smooth ODEs is determined by the sign of real part of eigenvalues of the Jacobian matrix. These eigenvalues are often referred to as the 'eigenvalues of the equilibrium'. The Jacobian Matrix of a system of smooth ODEs is the matrix of the partial derivatives of the right-hand side with respect to state variables

$$J = D_x f = f_x = \left(\frac{\partial f_i}{\partial x_j} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

where all derivatives are evaluated at the equilibrium point. Its eigenvalues determine linear stability properties of the equilibrium.

An equilibrium is asymptotically stable if all eigenvalues have negative real parts; it is unstable if at least one eigenvalue has positive real part.

Hyperbolic Equilibrium: Hartman-Grobman Theorem

In mathematics, in the study of dynamical systems, the **Hartman-Grobman theorem** or linearization theorem is an important theorem about the local behavior of dynamical systems in the neighborhood of a hyperbolic fixed point.

Basically the theorem states that the behavior of a dynamical system near a hyperbolic fixed point is qualitatively the same as the behavior of its linearization near the origin. Therefore when dealing with such fixed points we can use the simpler linearization of the system to analyze its behavior.

Let $f:\Re^n \to \Re^n$ be a smooth map with a hyperbolic fixed point *p*.
Let *A* denote the linearization of *f* at point *p*.
Then there exist a neighborhood *U* of *p* and a homeomorphism $h:U \to \Re^n$ such that

$$f_u = h^{-1}Ah$$

that is in a neighborhood *U* of *p*, f is topologically conjugate to its linearization.

In general, even for infinitely differentiable maps f, the homomorphism h need not to be smooth, nor even locally Lipschitz. However, it turns out to be Hölder continuous, with an exponent depending on the constant of hyperbolicity of A.

Hartman, Philip (August 1960). "A lemma in the theory of structural stability of differential equations". Proc. A.M.S. 11 (4): 610–620. doi:10.2307/2034720.

Hyperbolic Equilibrium: Hartman-Grobman Theorem for differential equations

local phase portrait of a hyperbolic equilibrium of a non-linear system is equivalent to that of its linearization. This statement has a mathematically precise form known as the **Hartman-Grobman Theorem**. It says that solutions of

$$x' = f(x)$$

in a small neighborhood of a hyperbolic equilibrium can be mapped with a homeomorphism (i.e., continuous map with a continuous inverse) onto solutions of the linear system

$$y' = Jy$$

where is the Jacobian matrix at the equilibrium. One says that these systems are locally topologically conjugate (equivalent). That is, adding nonlinear terms to a linear system at a hyperbolic equilibrium may distort but does not change qualitatively the phase portrait near the equilibrium.

If at least **one eigenvalue of the Jacobian matrix is zero or has zero real part**, then the equilibrium is said to be non-hyperbolic. **Non-hyperbolic equilibria are not robust (i.e., the system is not structurally stable):** Small perturbations can result in a local bifurcation of a non-hyperbolic equilibrium, i.e., it can change stability, disappear, or split into many equilibria. Some refer to such an equilibrium by the name of the bifurcation, e.g., saddle-node equilibrium.

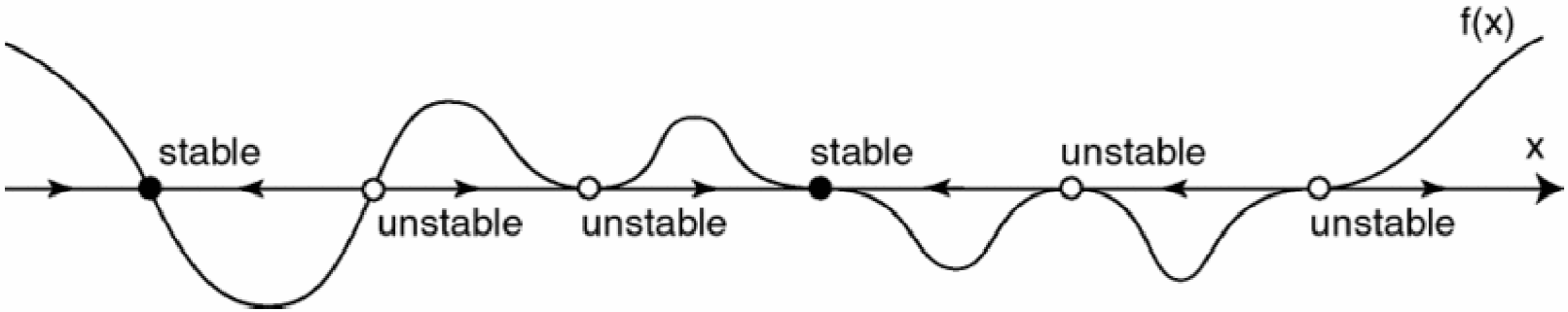
In practice, one often has to consider **non-hyperbolic equilibria with all eigenvalues having negative or zero real parts.** These equilibria are sometimes referred to as being critical. Their stability cannot be determined from the signs of the eigenvalues of the Jacobian matrix; it depends on the nonlinear terms of *f*.

Types of Equilibrium: One-Dimensional Space

Consider a one-dimensional (scalar) dynamical system

$$x' = f(x), \quad x \in \Re^1$$

$$J = f'(x)$$



Equilibria of a one-dimensional system

Its equilibria are the zeros of the function *f(x)*. An equilibrium is asymptotically stable when *f'(x)<0*. that is, the slope of *f* is negative. It is unstable when *f'(x)>0*. The left two equilibria in the figure are hyperbolic (f'(x)=0), the others are non-hyperbolic because the slope (eigenvalue) is zero. Nevertheless, a non-hyperbolic equilibrium of a one-dimensional system is stable if the function changes the sign from positive to negative at the equilibrium.

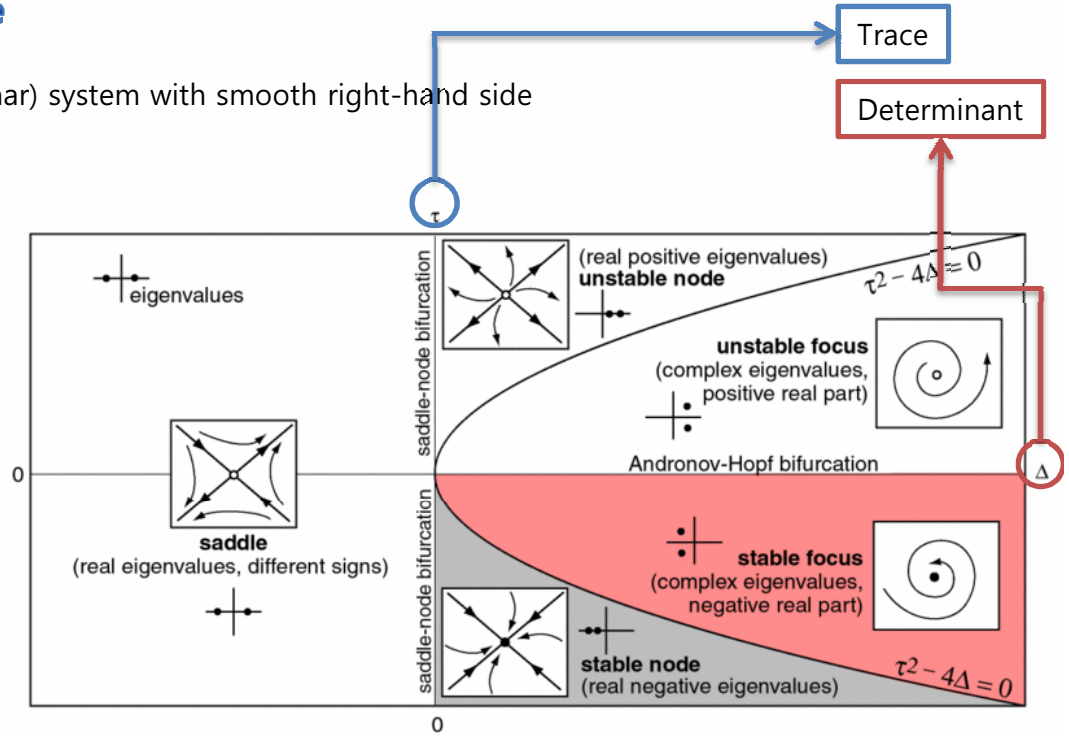
Types of Equilibrium: Two-Dimensional Space

Consider a two-dimensional (planar) system with smooth right-hand side

$$x_1' = f_1(x_1, x_2)$$
$$x_2' = f_2(x_1, x_2)$$

The Jacobian matrix has the form

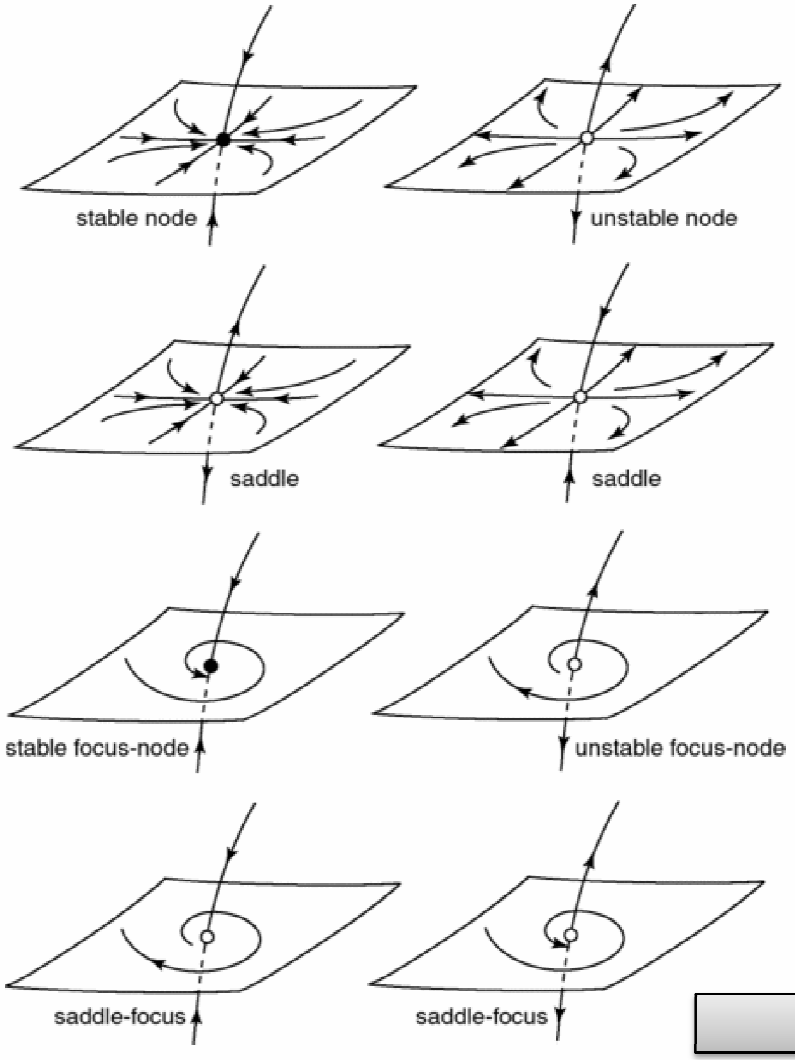
$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$



It has two eigenvalues, which are either both real or complex-conjugate. A hyperbolic equilibrium can be a

- **Node** when both **eigenvalues are real and of the same sign**. The node is stable when the eigenvalues are negative and unstable when they are positive. For the stable node, the eigenvalue(s) with minimal absolute value of the real part is called principle or leading; when the eigenvalues are different, all orbits but two tend to the node along the leading eigenvector (the picture is reversed for the unstable node);
- **Saddle** when **eigenvalues are real and of opposite signs.** The saddle is always unstable;
- **Focus** (sometimes called spiral point) when **eigenvalues are complex-conjugate;** The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.

Types of Equilibrium: Three-Dimensional Space

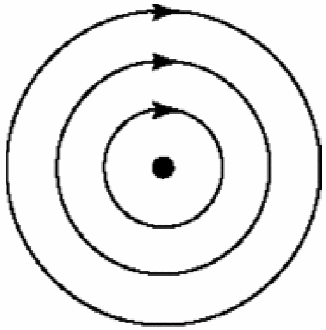


The Jacobian matrix of a three-dimensional system has 3 eigenvalues, one of which must be real and the other two can be either both real or complex-conjugate. A hyperbolic equilibrium can be

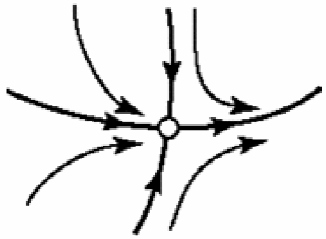
- **Node** when all eigenvalues are real and have the same sign; The node is stable (unstable) when the eigenvalues are negative (positive);
- **Saddle** when all eigenvalues are real and at least one of them is positive and at least one is negative; Saddles are always unstable;
- **Focus-Node** when it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign; The equilibrium is stable (unstable) when the sign is negative (positive);
- **Saddle-Focus** when it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues; This type of equilibrium is always unstable.

Examples of equilibria in R^3

Nonhyperbolic Equilibrium



center



saddle-node

Bogdanov-Takens

There are many more types of non-hyperbolic equilibria, i.e., those that have at least one eigenvalue with zero real part, since the phase portrait in a small neighborhood of such equilibria also depends on the nonlinear terms of *f(x)*. Most of these equilibria do not have names or are named after the type of the bifurcation in which they play a role.

The **center equilibrium** occurs when a system has only two eigenvalues on the imaginary axis, namely, one pair of pure-imaginary eigenvalues. Centers in linear systems have families of concentric periodic orbits around them, as in the figure. Many refer to centers only in the context of two-dimensional systems or Hamiltonian systems. If all other eigenvalues have negative real parts, centers are neutrally stable but not asymptotically stable. A pair of pure-imaginary eigenvalues also occurs in Andronov-Hopf bifurcation, however due to the nonlinear terms, the neighborhood of such an equilibrium looks like a focus; it could be asymptotically stable (supercritical Andronov-Hopf bifurcation) or unstable (subcritical Andronov-Hopf bifurcation) even if the other eigenvalues have negative real parts.

The **saddle-node equilibrium** occurs in nonlinear systems with one zero eigenvalue when the system undergoes the saddle-node bifurcation, where a saddle and a node approach each other, coalesce into a single equilibrium (depicted in the figure), and then disappear. Saddle-nodes are always unstable.

The **Bogdanov-Takens equilibrium** occurs in nonlinear systems with 2 zero eigenvalues, typically when the system undergoes the Bogdanov-Takens bifurcation. It is also an unstable equilibrium.

Examples of non-hyperbolic equilibria in R^2

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