Linear Preserver Problems
and their Solutions,
Problems, Conjectures, etc.

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Let $\mathcal{G}_n$ denote the set of all simple loopless undirected graphs on $n$ vertices. Let $s_n : \mathcal{G}_n \rightarrow \mathcal{B}^n$ be a mapping such that for $G \in \mathcal{G}_n$, the $i^{th}$ coordinate of $s_n(G)$, $s_n(G)_i$, is one if $G$ contains an $i$-cycle, and zero otherwise. The mapping $s_n$ is called a cycle sequence mapping and $s_n(G)$ is called the cycle sequence of $G$. Let $e_{i,j}$ denote the edge joining vertices $i$ and $j$. We also let $e_{i,j}$ denote the graph consisting of the one edge $e_{i,j}$. Let $+$ denote the union of two graphs. Let $T : \mathcal{G}_n \rightarrow \mathcal{G}_n$ be a linear operator that preserves the mapping $s_n$.

**Theorem 0.1** For $n \geq 4$, $T$ is a vertex permutation.

**Proof.** Suppose $T(X) = O$ for some $X \in \mathcal{G}_n$. then there is some pair $(i, j)$ such
that $T(e_{i,j}) = 0$. (Due to the Boolean arithmetic used in the semiring.) Without loss of generality we may assume that $T(e_{1,2}) = 0$. Now, $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ must have at least one $n$-cycle and cycles of no other length. But $e_{2,3} + \cdots + e_{n-1,n} + e_{n,1}$ has no cycles, so $T(e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ has no cycle, but $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1}) = T(e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$, a contradiction. Thus $T$ is nonsingular.

Suppose that $|T(e_{i,j})| \geq 2$, without loss of generality, $(i,j) = (1,2)$. Then $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ must contain only $n$-cycles. But, then, $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ has exactly $n$ edges. So, there is some $(i, i + 1)$ such that

$$T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1} \setminus e_{i,i+1}) = T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1}).$$

But then,
\[ T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1} \setminus e_{i,i+1}) \]

has no cycles while

\[ T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1}) \]

has an \( n \) cycle, a contradiction. Thus, \( T \) permutes the edges.

Now, consider \( T(e_{1,2} + e_{1,3} + e_{1,4}) \). If this is not a star, since it must be cycle free, it is either a) three disjoint edges, i.e. \( e_{1,2} + e_{3,4} + e_{5,6} \), b) a two path and a disjoint edge, i.e. \( (e_{1,2} + e_{2,3}) + e_{4,5} \), or c) a three path, i.e. \( e_{1,2} + e_{2,3} + e_{3,4} \). By considering the possible images, \( T(e_{2,3}), T(e_{3,4}) \), and \( T(e_{4,1}) \) one obtains a contradiction, since either some image of a three-cycle will not be a three-cycle or the image of the tree-star and one other edge will be a four-cycle. It follows that \( T \) maps stars to stars.

Thus, \( T \) is bijective and preserves stars. Thus \( T \) is a vertex permutation. See, e.g. Beasley, Pullman [?].
Further investigations.

1. $T$ preserves the set of cycles of length 3 and the set of cycles of length $k$ for some $4 \leq k \leq n$.

2. $T$ preserves the set of cycles of length $n$ and the set of cycles of length $k$ for some $3 \leq k \leq n - 1$.

3. $T$ preserves the set of cycles of length $i$ and the set of cycles of length $j$ for some $3 \leq i < j \leq n$.

4. $T$ is invertible and $T$ preserves the set of cycles of length $k$ for some $3 \leq k \leq n$.

5. $T$ strongly preserves the set of cycles of length $k$ for some $3 \leq k \leq n$. 
6. \( T(K) = K \) and preserves the set of cycles of length \( k \) for some \( 4 \leq k \leq n \) where \( K \) is the complete graph on \( n \) vertices.

**Also** The above extended to the set of directed graphs. Nearly the same proof works for the cycle sequence mapping.