Linear Preserver Problems and their Solutions, Problems, Conjectures, etc.

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Cycle sequence preservers

Let $\mathcal{G}_n$ denote the set of all simple loopless undirected graphs on n vertices. Let $s_n : \mathcal{G}_n \to \mathcal{B}^n$ be a mapping such that for $G \in \mathcal{G}_n$, the i^{th} coordinate of $s_n(G)$, $s_n(G)_i$, is one if G contains an i-cycle, and zero otherwise. The mapping s_n is called a *cycle sequence* mapping and $s_n(G)$ is called the *cycle sequence of* G. Let $e_{i,j}$ denote the edge joining vertices i and j. We also let $e_{i,j}$ denote the graph consisting of the one edge $e_{i,j}$. Let $+$ denote the union of two graphs. Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves the mapping s_n.

Theorem 0.1 *For $n \geq 4$, T is a vertex permutation.*

Proof. Suppose $T(X) = O$ for some $X \in \mathcal{G}_n$. then there is some pair (i, j) such that $T(e_{i,j}) = 0$. (Due to the Boolean arithmetic used in the semiring.) Without loss of generality we may assume that $T(e_{1,2}) = 0$. Now, $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ must have at least one n-cycle and cycles of no other length. But $e_{2,3} + \cdots + e_{n-1,n} + e_{n,1}$ has no cycles, so $T(e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ has no cycle, but $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1}) = T(e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$, a contradiction. Thus T is nonsingular.

Suppose that $|T(e_{i,j})| \geq 2$, without loss of generality, $(i,j) = (1,2)$. Then $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ must contain only n-cycles. But, then, $T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ has exactly n edges. So, there is some $(i, i+1)$ such that

$T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1} \setminus e_{i,i+1}) = T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$. But then,

$T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1} \setminus e_{i,i+1})$ has no cycles while

$T(e_{1,2} + e_{2,3} + \cdots + e_{n-1,n} + e_{n,1})$ has an n cycle, a contradiction. Thus, T permutes the edges.

Now, consider $T(e_{1,2} + e_{1,3} + e_{1,4})$. If this is not a star, since it must be cycle free, it is either a) three disjoint edges, i.e. $e_{1,2} + e_{3,4} + e_{5,6}$, b) a two path and a disjoint edge, i.e. $(e_{1,2} + e_{2,3}) + e_{4,5}$, or c) a three path, i.e. $e_{1,2} + e_{2,3} + e_{3,4}$. By considering the possible images, $T(e_{2,3}), T(e_{3,4})$, and $T(e_{4,1})$ one obtains a contradiction, since either some image of a three-cycle will not be a three-cycle or the image of the tree-star and one other edge will be a four-cycle. It follows that T maps stars to stars.

Thus, T is bijective and preserves stars. Thus T is a vertex permutation. See, e.g. Beasley, Pullman [**?**]. ∎

Further investigations.

1. T preserves the set of cycles of length 3 and the set of cycles of length k for some $4 \leq k \leq n$.

2. T preserves the set of cycles of length n and the set of cycles of length k for some $3 \leq k \leq n-1$.

3. T preserves the set of cycles of length i and the set of cycles of length j for some $3 \leq i < j \leq n$.

4. T is invertible and T preserves the set of cycles of length k for some $3 \leq k \leq n$.

5. T strongly preserves the set of cycles of length k for some $3 \leq k \leq n$.

6. $T(K) = K$ and preserves the set of cycles of length k for some $4 \leq k \leq n$ where K is the complete graph on n vertices.

ALSO The above extended to the set of directed graphs. Nearly the same proof works for the cycle sequence mapping.