Linear Preserver Problems and their Solutions, Problems, Conjectures, etc.

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Sets of primitive matrices defined by exponents

Define: Cell, primitive, exponent, star, double star (ie, the double star centered at vertex 1 is $E_{1,1}+\sum_{i=2}^{n}(E_{1,i}+E_{i,1})$), etc.

Theorem 0.1 *Let $T\mathcal{M}_n(\mathcal{B}) \to \mathcal{M}_n(\mathcal{B})$ preserve the set of primitive matrices of exponent 1 and the set of primitive matrices of exponent 2. Then T is a (P, P^t)-operator for some permutation matrix P.*

Proof. Since T preserves exponent 1, $T(J) = J$ as J is the only matrix of exponent 1. If $T(X) = O$ for some X then $T(E_{i,j}) = O$ for some i, j. Let $A = J \setminus E_{i,j}$. Then $T(A) = T(J)$, and A has exponent 2 while J has exponent 1, a contradiction. Thus, T is nonsingular.

Suppose that the image of some cell, E, is not a cell. Then there exist at most n^2-2 cells $E_1,\cdots,E_{n^2-2}$ such that $T(E)+T(E_1+\cdots+E_{n^2-2})=J$, but $E+E_1+\cdots+E_{n^2-2}$ has exponent 2, a contradiction. Thus the image of a cell is a cell and since $T(J)=J$, T is bijective on the set of cells.

Now, the only matrices of exponent 2 with $2n-1$ nonzero entries are double stars, thus, T preserves double stars, and hence T is a (P,P^t)-operator for some permutation matrix P. ∎

Further investigations.

1. T preserves the set of primitive matrices of exponent i and the set of primitive matrices of exponent j for some $1 \leq i < j \leq n^2 - 2n + 2$.

2. T is invertible and T preserves the set of primitive matrices of exponent k for some $2 \leq k \leq n^2 - 2n + 2$.

3. T strongly preserves the set of primitive matrices of exponent k for some $2 \leq k \leq n^2 - 2n + 2$.

See Beasley and Pullman [**?**, **?**, **?**].