Preserving Regular Tournaments and Term Rank-1*

LeRoy B. Beasley† and David E. Brown‡
Department of Mathematics and Statistics, Utah State University
Logan, Utah 84322-3900, USA
and
Alexander E. Guterman§
Department of Algebra, Faculty of Mathematics and Mechanics,
Moscow State University, Moscow, 119991, GSP-1, Russia

Recall from last week:

Definition 0.1 *An operator* T *is called a* (P,Q)-operator *if there exist permutation matrices* P *and* Q *such that* $T(X) = PXQ$ *for all* $X \in \mathcal{M}_n(\mathbf{B})$*, or* $T(X) = PX^tQ$ *for all* $X \in \mathcal{M}_n(\mathbf{B})$*.*

Corollary 0.2 *Let* $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ *be an additive operator and* $T(0) = 0$*. If* T *preserves term rank-1 then either* T *is a* (P, P^t)*-operator, where* P *is a permutation matrix or* $T(A)$ *has a zero row or a zero column for all* $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$*.*

Regular Tournament Matrices and Preservers

In recent years there has been interest in tournaments and preserver problems.

Definition 1.3 *A* tournament *on n vertices is a directed graph which is an orientation of the complete graph on n vertices, that is a tournament is a loopless digraph in which any two distinct vertices are connected by exactly one arc.*

Definition 1.4 *A* regular tournament *on n vertices is a tournament which has the same number of outgoing arcs for any vertex.*

Remark 1.5 *Note that in this case in each vertex we have exactly $\frac{n-1}{2}$ outgoing arcs and, hence, n must be odd. Whence the following definition is appropriate.*

Definition 1.6 *Let n be an even number. A* nearly regular tournament *on n vertices is a tournament which has $\frac{n}{2}$ vertices with $\frac{n}{2}$ outgoing arcs and $\frac{n}{2}$ vertices with $\frac{n-2}{2}$ outgoing arcs.*

Definition 1.7 *An* adjacency *matrix of a digraph is a matrix $M = [m_{i,j}]$ such that $m_{i,j} = 1$ if and only if there is an arc with initial vertex i and terminal vertex j.*

Remark 1.8 *Let $A(T)$ be the adjacency matrix of the tournament T. Then, $A(T)$ is a $(0,1)$-matrix such that $A(T)+A(T)^t+I = J$, where I denotes the identity matrix and J the matrix of all ones. That is, if M is the adjacency matrix of a tournament digraph, M has a zero diagonal and for $i \neq j$, $m_{i,j} \neq 0$ if and only if $m_{j,i} = 0$.*

Definition 1.9 *Adjacency matrices of tournament digraphs are usually called* tournament matrices.

Definition 1.10 *A* digon *is a directed graph whose adjacency matrix is of the form* $E_{i,j}+E_{j,i}$, $i \neq j$, *that is a directed two-cycle.*

Remark 1.11 *If* n *is odd, then the adjacency matrices of regular tournament digraphs (*regular tournament matrices*) are tournament matrices with the same number of entries equal to 1 in each row. If* n *is even, the adjacency matrices of nearly regular tournament digraphs (*nearly regular tournament matrices*) are tournament matrices that have* $\frac{n}{2}$ *rows with* $\frac{n}{2}$ *entries equal to 1, and* $\frac{n}{2}$ *rows with* $\frac{n-2}{2}$ *entries equal to 1.*

When working with matrices that correspond to tournament digraphs we use Boolean arithmetics on matrix entries. This technique is common and advantageous.

In this section we apply the above results to the characterization of transformations preserving regular and nearly regular tournament matrices.

Let D be a directed graph with each pair of its n vertices connected by at most one arc; i.e., D is a subdigraph of a tournament. The next result has an interpretation that answers the following question: What is the maximum number of arcs that can be present in D before knowing whether D can be transformed into a regular or nearly regular tournament by adding arcs between nonadjacent vertices?

Let $r_i = r_i(A)$ denote the number of nonzero entries in the i^{th} row of A and $c_j = c_j(A)$ the number of nonzero entries in the j^{th} column of A.

In Beasley Brown and Reid, the following two theorems were proved.

Theorem 1.12 *Let n be odd and A be an $n \times n$ $(0,1)$-matrix dominated by a tournament matrix such that the row sums of A are $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq i \leq n$, if $r_i \leq \max(\frac{n-1}{2} - i + 1, 0)$, then A is dominated by a regular tournament matrix.*

Theorem 1.13 *Let n be even and A be a $(0,1)$ matrix dominated by a tournament matrix such that the row sums of A are $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq i \leq n$, if $r_i \leq \max(\frac{n}{2} - i + 1, 0)$, then A is dominated by a nearly regular tournament matrix.*

In the sequel we will use the following immediate corollary to Theorems 1.12 and 1.13.

Corollary 1.14 *Let n be an integer greater than 1, and A be an $n \times n$ $(0,1)$-matrix. If A is the sum of at most $\lceil \frac{n+1}{2} \rceil$ off-diagonal cells such that A does not dominate a digon* *and with no more than* $\lceil \frac{n-1}{2} \rceil$ *nonzero entries in any one row or column, then if* n *is odd (even), there is a regular (nearly regular) tournament matrix which dominates* A*.*

Theorem 1.15 *If* $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ *is an additive operator that preserves both term rank-1 and regular tournament matrices when* n *is odd or nearly regular tournament matrices when* n *is even, then* T *is a* (P, P^t)*-operator.*

Proof. The case $n = 2$ is trivial.

For $n = 3$ if $T(O) \neq O$, then $T(O)$ has term rank-1 and is dominated by a regular tournament. Say without loss of generality that $T(O) \leq E_{1,2}+E_{2,3}+E_{3,1}$. Then, without loss of generality we may assume that $T(O) \leq E_{1,2}$. Further, for some cell F, $T(F) \geq E_{2,3}$, so that $T(F) = T(F + O) \geq$

$E_{1,2} + E_{2,3}$, contradicting that T preserves term rank-1. Thus $T(O) = O$.

Suppose that $T(O) \neq O$ and $n \geq 4$. Then, since T preserves regular (nearly regular) tournaments and term rank-1, there is a row or column whose image has at least $\lceil \frac{n-1}{2} \rceil$ non-zero entries. This matrix must be of term rank-1. Say $|T(R_1^{(0)})| \geq \lceil \frac{n-1}{2} \rceil$ and $T(R_1^{(0)}) \leq L_j$ where L_j is either $R_j^{(0)}$ or $C_j^{(0)}$. It follows that $T(O) \leq L_j$ since the term rank of $(R_1^{(0)} + O)$ is 1. Also, there is some other row with the same properties, say $|T(R_i^{(0)})| \geq \lceil \frac{n-1}{2} \rceil$ and $T(R_i^{(0)}) \leq L_k$ where L_k is either $R_k^{(0)}$ or $C_k^{(0)}$. It now follows that $T(O) \leq L_k$ since the term rank of $(R_i^{(0)} + O)$ is 1. This can only happen if $L_j \leq R_j^{(0)}$ and $L_k \leq C_k^{(0)}$ and $T(O) = E_{j,k}$ or $L_j \leq C_j^{(0)}$ and $L_k \leq R_k^{(0)}$ and $T(O) = E_{k,j}$. In either case there is a cell, F, whose image dominates $E_{r,s}$

for some $r \neq j$ and $s \neq k$ (or $r \neq k$ and $s \neq j$). But then $T(O+F)$, which must have term rank-1, dominates $E_{j,k}+E_{r,s}$ (or $E_{k,j}+E_{r,s}$), contradicting that the term rank of $T(O+F)$ must be one.

By Corollary 0.2 we have that either T is a (P, P^t)-operator, where P is a permutation matrix, or $T(K)$ has a zero row or a zero column. If $T(K)$ has a zero row or a zero column, then T does not preserve regular tournaments as there are too many zeros in the image $T(K)$. Thus T is a (P, P^t)-operator. ∎

To prove our last statement we recall the following result which follows from Beasley and Guterman.

Lemma 1.16 *Let* $T : \mathcal{M}_n(\mathbf{B}) \to \mathcal{M}_n(\mathbf{B})$ *(respectively,* $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$*) be a linear operator. Then the following are equivalent:*

1. *T is bijective.*

2. *T is surjective.*

3. *T is injective.*

4. *There exists a permutation σ on $\{(i,j) \mid i,j = 1,2,\cdots,n\}$ (respectively, σ on $\{(i,j) \mid i,j = 1,2,\cdots,n;\ i \neq j\}$) such that $T(E_{i,j}) = E_{\sigma(i,j)}$.*

Proof. Since $\mathcal{M}_n(\mathbf{B})$ and $\mathcal{M}_n(\mathbf{B})^{(0)}$ are finite sets, the result follows from Beasley and Guterman. ∎

Theorem 1.17 *If $n > 3$ and $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ is a surjective additive operator that preserves regular tournament matrices when n is odd or nearly regular tournament matrices when n is even, then T is a (P, P^t)-operator.*

Proof. By Lemma 1.16, T is bijective and we have that T strongly preserves regular (nearly regular) tournament matrices, and T maps cells to cells.

Suppose that n is even and T does not preserve term rank-1. Then since T is bijective, there are two noncollinear cells whose images are collinear. Say, without loss of generality, that $T(E_{i,j}) = E_{r,s}$ and $T(E_{k,l}) = E_{r,t}$ with $i \neq k$ and $j \neq l$ (all other cases are similar).

If $(k,l) = (j,i)$ then there are $\frac{n^2-n-4}{2}$ cells $G_1, G_2, \cdots, G_{\frac{n^2-n-4}{2}}$ such that $G_1 + G_2 + \cdots + G_{\frac{n^2-n-4}{2}} + E_{r,s} + E_{r,t}$ is a nearly regular tournament, but if $F_i = T^{-1}(G_i)$, $F_1 + F_2 + \cdots + F_{\frac{n^2-n-4}{2}} + E_{i,j} + E_{j,i}$ is not a nearly regular tournament since it dominates a digon, contradicting that T strongly preserves nearly regular tournament matrices.

Thus, the image of a digon is never the sum of two collinear cells.

Let $F_1, F_2, \cdots, F_q$ be cells distinct from $E_{i,j}, E_{j,i}, E_{l,k}$, and $E_{k,l}$ whose images are in row r where $q = \frac{n-2}{2}$. Let $A = F_1 + F_2 + \cdots + F_q + E_{i,j} + E_{k,l}$. Then, as above, A is dominated by a tournament matrix, and hence, by Corollary 1.14, A is dominated by a nearly regular tournament, M, but $T(A)$ has all $\frac{n+2}{2}$ nonzero entries in one row, and hence $T(M)$ cannot be a nearly regular tournament matrix, a contradiction. That T preserves term rank-1 and the theorem follows from Theorem 1.15.

Arguing parallel to the even case, suppose that $n \geq 5$ is odd and T does not preserve term rank-1. Then since T is bijective, there are two noncollinear cells whose images are collinear. Say, without loss of generality, that $T(E_{i,j}) = E_{r,s}$ and $T(E_{k,l}) =$

$E_{r,t}$ with $i \neq k$ and $j \neq l$ (all other cases are similar).

If $(k,l) = (j,i)$ then there are $\frac{n^2-n-4}{2}$ cells $G_1, G_2, \cdots, G_{\frac{n^2-n-4}{2}}$ such that $G_1 + G_2 + \cdots + G_{\frac{n^2-n-4}{2}} + E_{r,s} + E_{r,t}$ is a regular tournament, but if $F_i = T^{-1}(G_i)$, $F_1 + F_2 + \cdots + F_{\frac{n^2-n-4}{2}} + E_{i,j} + E_{j,i}$ is not a regular tournament since it dominates a digon, contradicting that T strongly preserves regular tournament matrices. Thus, the image of a digon is never the sum of two collinear cells.

Let $F_1, F_2, \cdots, F_q$ be cells distinct from $E_{i,j}, E_{j,i}, E_{l,k}$, and $E_{k,l}$ whose images are in row r where $q = \frac{n-3}{2}$. Note that since T strongly preserves regular tournaments the set $\{F_1, F_2, \cdots, F_q, E_{i,j}, E_{k,l}\}$ has no digon pairs, otherwise, since T is bijective the image of the sum of those two cells is dominated by a regular tournament, but they cannot be, otherwise T would map a non-tournament to a regular tournament. Let $A = F_1 + F_2 + \cdots + F_q + E_{i,j} + E_{k,l}$. Then, by Corollary 1.14, A is dominated by a regular tournament, M, but $T(A)$ has all $\frac{n+1}{2}$ nonzero entries in one row, and hence, $T(M)$ cannot be a regular tournament matrix, a contradiction. That is T preserves term rank-1 and the theorem follows from Theorem 1.15.

∎

Remark 1.18 *Let us note that if $n = 2$ then Theorem 1.17 holds trivially, with $P = I$.*

Remark 1.19 *Theorem 1.17 is not true for $n = 3$ since the operator*

$$T : \mathcal{M}_3(\mathbf{B})^{(0)} \to \mathcal{M}_3(\mathbf{B})^{(0)}$$

defined by

$$T\left(\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & c \\ b & 0 & e \\ d & f & 0 \end{bmatrix}$$

preserves regular tournaments but is not a (P, P^t)-operator since the image of every digon is the sum of two collinear cells.