Linear Preserver Problems and their Solutions, Matrices over Fields.

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There are two forms of preserver problems, one fixes a function, set, or relation and asks what linear transformations leave it invariant. This is the most common form studied. The other form fixes the structure of a linear transformation and asks what functions, sets, and relations are fixed by it. The first form of problem is well studied, the second is not as easy and few full characterizations are known. We consider the first form first.

1. Introduction

The study of linear and additive operators that leave functions or relations on matrices invariant has been an active research area in the past half century. In this talk we will present some of the history and results appearing in this area of endeavor. This talk is not intended to be all inclusive. Indeed, probably no article could encompass all the relevant research. Here we restrict ourselves to matrices whose entries are in a field and do not discuss some large areas of preserver research, for example, there is little discussion of preservers of numerical range, numerical radius, sets defined by eigenvalues, sets defined by singular values, norms, etc. These should be addressed by another author in due time.

Let $T : \mathcal{M}_{m,n}(\mathbb{C}) \to \mathcal{M}_{m,n}(\mathbb{C})$ be a linear operator such that for any $A \in \mathcal{M}_{m,n}(\mathbb{C})$ $\det(T(A)) = \det(A)$. What can be said of the structure of T? This was the first question considered in the study of what is now known as linear preserver problems.

Let $\mathcal{K}$ be a linear space of matrices over some field $\mathbb{F}$. Some possible examples include $\mathcal{M}_{m,n}(\mathbb{C})$, $\mathcal{M}_{m,n}(\mathbb{R})$,
$\{U \in \mathcal{M}_{m,n}(\mathbb{C}) | U \text{ is upper triangular}\}$,
$\{A \in \mathcal{M}_{m,n}(\mathbb{C}) | \text{tr}(A) = 0\}$, etc. Sometimes the entries in the matrices in $\mathcal{K}$ need not be in $\mathbf{F}$, for example we may consider $\mathcal{H} = \{H \in \mathcal{M}_{m,n}(\mathbb{C}) | H = H^*\}$, i.e., the set of all Hermitian matrices, as a linear space over $\mathbb{R}$ since $\mathcal{H}$ is not a vector space over $\mathbb{C}$ while it is over $\mathbb{R}$. To see this one only needs observe that $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \in \mathcal{H}$

while $i\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \notin \mathcal{H}$.

There are three basic types of preserver problems:

1. Given a function $f : \mathcal{K} \to \mathcal{S}$ determine the structure of all linear operators $T : \mathcal{K} \to \mathcal{K}$ such that $f(T(A)) = f(A)$ for all $A \in \mathcal{K}$. (Note Here S can be any set and f can be any type of function desired.)

2. Let $\mathcal{S}$ be a subset of $\mathcal{K}$. Determine the structure of all linear operators $T : \mathcal{K} \to \mathcal{K}$ such that $A \in \mathcal{S}$ implies that $T(A) \in \mathcal{S}$.

3. Let "$\sim$" be a relation of $\mathcal{K}$. Determine the structure of all linear operators T :

$\mathcal{K} \to \mathcal{K}$ such that $A \sim B$ implies that $T(A) \sim T(B)$.

It may be noted that some preserver problems can be stated in more than one of the types above.

In this article, we first present some of the history of the preserver problem and proceed to present some of the present endeavors. The work done by Marvin Marcus and his students and colleagues in the late 1950's was a turning point in linear preserver problem research. Thus, Section 3 will be a review of the research done leading up to the work of Marcus. Section 4 will give the work of Marcus and his students and colleagues during the 1950's and 1960's.

2. Preliminaries

Some notation and definitions that will be used throughout this article are given here. Some commonly used terms are not defined. Their definition may be found in any of the standard texts on linear algebra or matrix theory. Let A be a matrix. We shall use the following notation for some familiar notions: The determinant of A, det(A); the permanent of A, per (A); the rank of A, rank(A); the trace of A, tr(A); the number of combination of n things taken r at a time, $\begin{pmatrix} n \\ r \end{pmatrix}$; the $m \times n$ matrix of all zero entries, $O_{m,n}$ (or O if the order is obvious form the context); the $m \times n$ matrix of all entries equal one, $J_{m,n}$ (or J if the order is obvious form the context); the set of all $m \times n$ matrices over a set (usually a field) $\mathbb{S}$, $\mathcal{M}_{m,n}(\mathbb{S})$; or if $m = n$,

$\mathcal{M}_n(\mathbb{S})$. We also use the notation $E_{i,j}$ to denote the matrix of whose (i,j) entry is one and all others are zero. We denote the transpose of A by A^t.

Let $\mathbb{S}$ be an algebraic set (multiplication and addition are defined) and let $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$ defined by $T(A) = UAV$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$ or by $T(A) = UA^tV$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$, where U and V are matrices in $\mathcal{M}_m(\mathbb{S})$ and $\mathcal{M}_n(\mathbb{S})$ respectively. We call T a (U,V)-*operator*. If U and or V have additional properties we add that to the notation, i.e., if $U = P$ and $V = P^t$ are permutation matrices we say that T is a (P,P^t)-operator.

3. The Early Years

The first results appearing on preserver problems were in 1897 by S. Kantor and G. Frobenius. They considered the preservers of the determinant function for various subspaces of $\mathcal{M}_n(\mathbb{C})$ and $\mathcal{M}_n(\mathbb{R})$: Let

1. $\mathcal{K} = \mathcal{M}_n(\mathbb{C})$,

2. $\mathcal{K}$ be the set of all symmetric matrices in $\mathcal{M}_n(\mathbb{R})$,

3. $\mathcal{K}$ be the set of all skew-symmetric matrices in $\mathcal{M}_n(\mathbb{R})$, or

4. $\mathcal{K}$ be the set of all matrices in $\mathcal{M}_n(\mathbb{C})$ whose trace is 0.

Then if $T : \mathcal{K} \to \mathcal{K}$ preserves the determinant, ($\det(T(X)) = \det(X)$) then T is a (U, V)-operator where

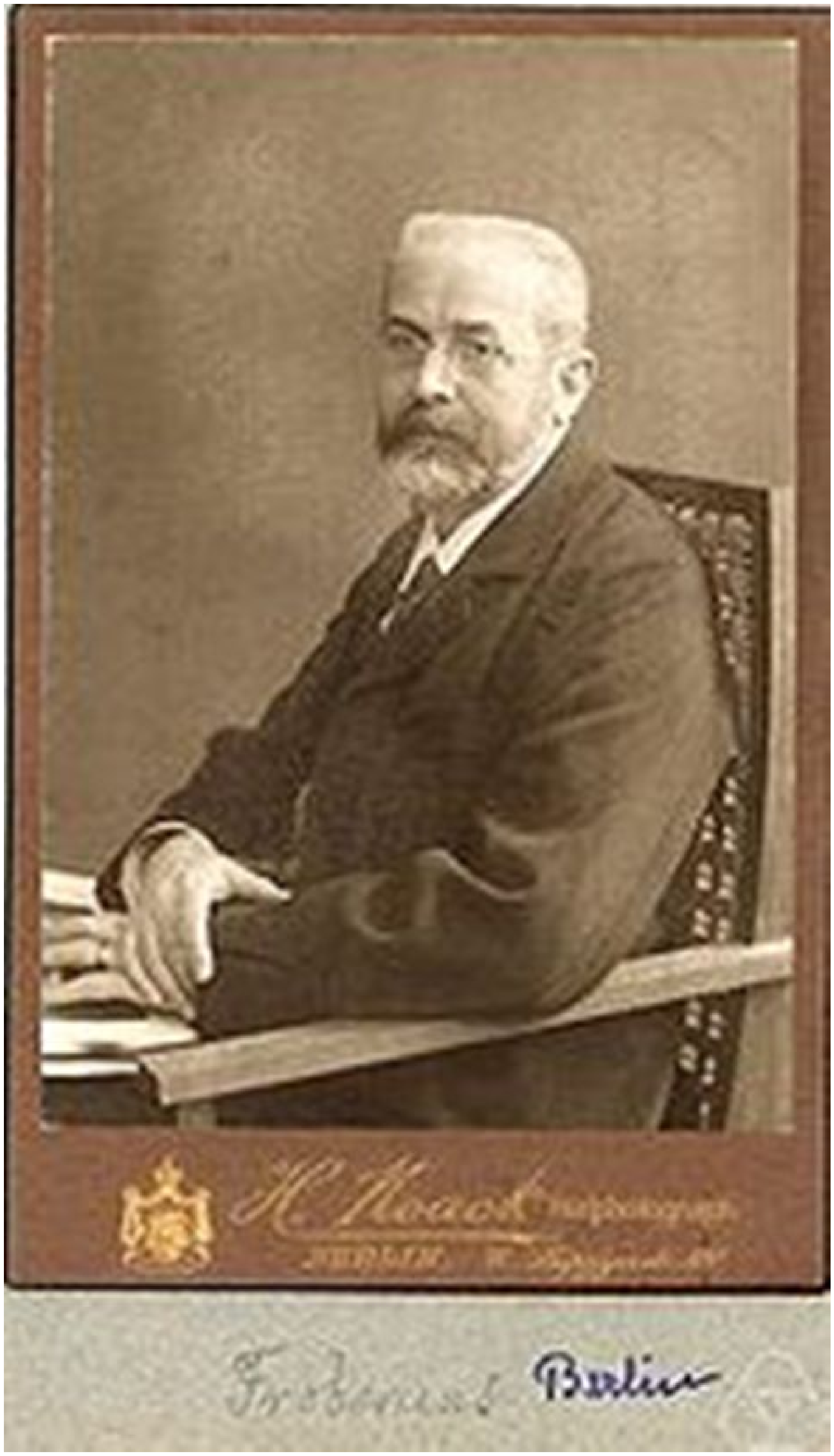
1. $\det(UV) = 1$,

2. $U = \xi A$, and $V = A^t$, where $\det(A) = 1$ and $\xi^n = 1$,

3. $U = \xi A$, and $V = A^t$, where $\det(A) = 1$ and $\xi^n = 1$, or

4. $U = \xi A$, and $V = A^{-1}$, where $\xi^n = 1$,

respectively.



Ferdinand Georg Frobenius 1849-1917

In the next half century there were less than a dozen researchers dealing with preserver problems. We will give some of the results that appeared in the first half of the last century.

1. Let $A \circ B$ denote the Hadamard (or Schur) product of two matrices, that is, for $A = (a_{i,j})$ $B = (b_{i,j})$ $A \circ B = C$ where $c_{i,j} = a_{i,j} b_{i,j}$. In 1913, G. Polya considered the problem of finding a mapping $T : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ such that $T(A) = A \circ B$ where B is some matrix, all of whose entries are either 1 or -1, and such that det(A) = per (T(A)). He showed that unless $n = 1$, or 2, there is no such T (the matrix B cannot be found).

George Polya 1887-1985

2. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $2 \leq r \leq n$. Let α and β be increasing sequences of $\{1, 2, \cdots, n\}$ of length r. We let $A[\alpha|\beta]$ be the $r \times r$ submatrix of A on rows in α and columns in β, . Let $C_r(A)$ denote the $\begin{pmatrix} n \\ r \end{pmatrix} \times \begin{pmatrix} n \\ r \end{pmatrix}$ matrix consisting of all the determinants of $A[\alpha|\beta]$ in doubly lexicographic order. The matrix $C_r(A)$ is called the r^{th} *compound matrix of* A. In 1925, I. Schur proved that for $T : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ a linear operator, and $S : \mathcal{M}_{\begin{pmatrix} n \\ r \end{pmatrix}, \begin{pmatrix} n \\ r \end{pmatrix}}(\mathbb{C}) \to \mathcal{M}_{\begin{pmatrix} n \\ r \end{pmatrix}, \begin{pmatrix} n \\ r \end{pmatrix}}(\mathbb{C})$ a nonsingular linear operator then, if $C_r(T(A)) = S(C_r(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$ then T is a (U, V)-operator.

Issai Schur 1875-1941

3. Between 1940 and the early 1950's six researchers, L. K. Hua, H. Jacob. N. Jacobson, R.V. Kadison, K. Morita , and M. Sugawara investigated the geometric basis of tensor spaces and the preservers of norms on various spaces. These results have proven to be the basis for norm preservers and for preservers involving singular values. Hua and Jacob investigated the preservers of covariance, the rank of the difference of two matrices. Their investigations showed that if T preserves rank then T is a (U,V)-operator.

4. Let $\mathcal{U}$ be a finite dimensional vector space over a field $\mathbb{F}$. Let $P_r(\mathcal{U})$ be the set of all subspaces of $\mathcal{U}$ of dimension $r, 1 < r < n$. Let $\mathcal{K}, \mathcal{L} \in P_r(\mathcal{U})$.

We say that $\mathcal{K}$ and $\mathcal{L}$ are *adjacent* if $\dim(\mathcal{K} \cap \mathcal{L}) = r - 1$. In 1949, W. L. Chow characterized all bijective (invertible) linear operators of $P_r(\mathcal{U})$ which preserve adjacency. His work was instrumental in the investigation of preservers of decomposable tensors.

5. In 1949, J. Dieudonné proved that every invertible linear operator on $\mathcal{M}_n(\mathbb{C})$ which preserved the set of singular matrices ($\det(A) = 0 \Rightarrow \det(T(A)) = 0$) is a (U, V)-operator for U and V nonsingular. In accomplishing this he showed that the variety of all singular matrices was of dimension $n^2 - n$ and any ideal of dimension $n^2 - n$ of singular matrices was a maximal ideal in $\mathcal{M}_n(\mathbb{F})$.

4. The Marcus Era

In the late 1950's and early 1960's, Marvin Marcus, his colleagues and students laid the ground work for the investigation of linear preserver problems that has continued until this day. Most of the later work on linear preservers depend on this early work of the Marcus group.

In 1957, Roger Purves wrote his Masters's thesis at the University of British Columbia entitled "Linear Transformations on Matrices" under the direction of Marvin Marcus.. This research led to the publication of the paper by Marcus and Purves concerning the preservers of the elementary symmetric functions.

Let $A \in \mathcal{M}_n(\mathbb{C})$ have eigenvalues $\lambda_1.\lambda_2.\cdots,\lambda_n$. For $1 \leq r \leq n$. let $E_r(A)$ be the sum of the product of the eigenvalues taken r at a time. That is, $E_r(A) = \sum_{\sigma \in \mathfrak{S}_r} \prod_{i=1}^{r} \lambda_{\sigma(i)}$ where $\mathfrak{S}_r$ is the set of all one-to-one mappings $\sigma : \{1, 2, \cdots, r\} \to \{1, 2, \cdots, n\}$. Equivalently, $E_r(A) = \mathrm{tr}(C_r(A))$ where $C_r(A)$ is the r^{th} compound matrix described above and researched by I. Schur in 1925.

It is easily seen that $E_1(A) = \mathrm{tr}(A)$ and $E_n(A) = \det(A)$.

Marcus and Purves proved:

Theorem 4.1 *Let* $A \in \mathcal{M}_n(\mathbb{C})$, $4 \leq r \leq n$, *and* $T : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ *be a linear transformation. Then,* $E_r(T(A)) = E_r(A)$ *for all* $A \in \mathcal{M}_n(\mathbb{C})$ *if and only if there exists a nonsingular matrix* S *and an* r^{th} *root of unity* θ *such that either* $T(X) = \theta S^{-1}XS$ *for all* $X \in \mathcal{M}_n(\mathbb{C})$ *or* $T(X) = \theta S^{-1}X^tS$ *for all* $X \in \mathcal{M}_n(\mathbb{C})$*. That is, within an* r^{th} *root of unity,* T *is a similarity operator.*

Note that if $r = n$ this result is the result of Frobenius and Kantor from 1897. Also, there exist linear operators that preserve E_1 and E_2 that are not (U,V)-operators: Let $T_1 : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ such that $T(E_{i,i}) = E_{i,i}$ and for $i \neq j$, $T(E_{i,j}) = B_{i,j}$ be any matrix such that $\mathrm{tr}(B_{i,j}) = 0$. Then T_1 preserves E_1.

Let $T_2 : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ such that $T(E(i,i) = E_{i,i}$ and for $i < j$, $T(E_{i,j} + E_{j,i}) = \alpha_{i,j}E_{i,j} + \beta_{i,j}E_{j,i}$ for any $\alpha_{i,j}$ and $\beta_{i,j}$ where $\alpha_{i,j}\beta_{i,j} = 1$. Then T_2 preserves E_2. In general, neither T_1 nor T_2 are (U,V)-operators.

Let $U \in \mathcal{M}_n(\mathbb{C})$. Then, U is *unitary* if $U^*U = UU^* = I$, where U^* is the conjugate transpose of U. The set $\mathfrak{U} = \{U \in \mathcal{M}_n(\mathbb{C}) : U \text{ is unitary}\}$, called the *unitary group*, is a subgroup of the n^{th} general linear group, $GL_n(\mathbb{C})$, the set of all nonsingular matrices in $\mathcal{M}_n(\mathbb{C})$. In 1959, Marcus proved that if T preserves the group $\mathfrak{U}$ then T is a unitary similarity operator, i. e., for some unitary matrix U, $T(X) = U^*XU$ for all $X \in \mathcal{M}_n(\mathbb{C})$ or $T(X) = U^*X^tU$ for all $X \in \mathcal{M}_n(\mathbb{C})$. This research gave rise to much of the work on preservers of various subgroups of $GL_n(\mathbb{C})$.

In 1959, Marcus and Moyls considered rank preserving linear operators on $\mathcal{M}_n(\mathbb{C})$ and reproved the results of Frobenius, Kantor, Hua and Jacob concerning preservers of the determinant and rank.

Also in 1959, Marcus and Moyls, proved that if $T : \mathcal{M}_{m,n}(\mathbb{C}) \to \mathcal{M}_{m,n}(\mathbb{C})$ preserves the set of all matrices of rank-1, then T is a (U, V)-operator with U and V nonsingular. This result was proved in the setting of tensor products. A rank one matrix can be associated with a tensor product of vectors thusly: If A is rank 1, then there are vectors $\mathbf{x}$ and $\mathbf{y}$ such that $A = \mathbf{x}\mathbf{y}^t$. This factoring of A is unique up to scalar multiples. That is, if rank(A) $= 1$ and $A = \mathbf{x}\mathbf{y}^t = \mathbf{u}\mathbf{v}^t$, then $\mathbf{x}$ is a multiple of $\mathbf{u}$ and $\mathbf{y}$ is a multiple of $\mathbf{v}$, since if the i^{th} column of A is nonzero, the i^{th} column of $A = \mathbf{x}y_i = \mathbf{u}v_i$ so that $\mathbf{x} = \frac{v_i}{y_i}\mathbf{u}$ and similarly,

$\mathbf{y} = \frac{u_i}{x_i}\mathbf{v}$. So that $A = \frac{v_i}{y_i}\mathbf{u}\frac{u_i}{x_i}\mathbf{v}$, etc. Note that $\frac{u_i}{x_i}\frac{v_i}{y_i} = 1$ Thus, associate with $\mathbf{x} \otimes \mathbf{y}$ the matrix $A = \mathbf{x}\mathbf{y}^t$. This association is a one-to one and onto mapping of $\mathbb{C}^m \otimes \mathbb{C}^n$ to $\mathcal{M}_{m,n}(\mathbb{C})$. (Here $\mathbb{C}^m \otimes \mathbb{C}^n$ denotes the set of all linear combinations of tensors of the form $\mathbf{x} \otimes \mathbf{y}$.)

Note: Marcus and many of his students researched linear preserver problems in the setting of tensor products, however, most of the research can be translated to linear preserver problems of matrices. The power of considering linear preservers of tensor products is that often the problem is reduced to considering inner products of the tensor spaces.

In 1960, Marcus and May revisited the work of I. Schur on compound matrices which resulted in several papers of Marcus et. al. on various forms of preservers of C_r.

In the next sections we summarize results obtained in the various types of preserver problems.

5. Linear Preserver results 1960-Present.

Most of the results concerning linear preservers are of the form: If T preserves $\mathcal{P}$ then T is a (U, V)-operator where *** Conditions on U and V ***. We highlight some of the results, emphasizing when T is not necessarily a (U, V)-operator.

5.1 Preservers of matrix functions.

Let $T : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ be a linear operator. The preservers of $\det, C_r, E_r, (r \geq 4)$, and rank have been characterized before 1960, and mentioned in the earlier sections. Since 1960, most of the preserver results have been to generalize the results already obtained to matrices over more general algebraic structures or to characterize the missing cases. Most of these characterizations are parallel to the earlier cases, for example, in 1970, Beasley showed that the preservers of E_3, the third elementary symmetric function followed the same characterization as do the preservers of $E_r, r \geq 4$. In 1976, Minc showed that if T preserves E_1 and det then T is a (V^{-1}, V)-operator, and if T preserves E_2 and det then T is a $(\epsilon V^{-1}, V)$-operator where $\epsilon = \pm 1$.

In 1960, Marcus and Westwick showed that for $\mathcal{H}$, the set of all $n \times n$ real skew-symmetric matrices, if $T : \mathcal{H} \to \mathcal{H}$ preserves E_{2r} then T is a $(U.V)$-operator, with the exception of the case $2r = 4$. In this case there is another class of preservers which can be seen by the following example: Let

$$T\left(\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}\right) =$$

$$\begin{bmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{bmatrix}$$

Then T preserves E_4(det) but is not a (U,V)-operator.

Let S_n denote the group of all permutations of $\{1, 2, \cdots, n\}$ and let $G \leq S_n$ be a subgroup. Let χ be a character of G. Let d_χ be a multilinear functional on $\mathcal{M}_n(\mathbb{C})$ defined by

$$\mathsf{d}_\chi(A) = \frac{1}{\chi(e)} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where e is the identity of S_n. The determinant is one of these functionals, the most well known. For the determinant, $G = S_n$ and $\chi = sgn$ the sign of the permutation. Another common functional of this type is the permanent function, per (A) $= \sum_{\sigma \in \mathsf{S_n}} \prod_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{a}_{\mathsf{i},\sigma(\mathsf{i})}$. Here $G = S_n$ and $\chi(\sigma) = 1$ for all σ.

In 1962, Marcus and May proved that if T preserves the permanent function, then T is a (U, V)-operator where $U = PD$ and $V = EQ$ where P and Q are permutation matrices and D and E are diagonal matrices with $\prod_{i=1}^{n} d_{i,i}e_{i,i} = 1$.

The characterization of linear preservers of other generalized matrix functions was by Botta where it was established that if G is a doubly transitive subgroup of S_n and χ is a linear character of G then, if T preserves d_χ then T is a (U, V)-operator where U is the product of a matrix whose entries depend on the character χ and a permutation matrix P, V is a permutation matrix and the permutation corresponding to PV is in G.

5.2 Preservers of sets of matrices.

In 1949, Dieudonné investigated nonsingular preservers of the set of singular matrices. We now give some results of preservers of other sets defined by functions or relations on matrices. We assume that $m \leq n$ and $\mathbb{F}$ is an algebraically closed field. Let $R_k = \{A \in \mathcal{M}_{m,n}(\mathbb{F}) : \text{rank}(A) = k\}$. Let $T : \mathcal{M}_{m,n}(\mathbb{F}) \to \mathcal{M}_{m,n}(\mathbb{F})$ be a linear operator such that if $A \in R_k$ then $T(A) \in R_k$, i.e., T preserves rank-k matrices. For $k = 1$ these were characterized in 1959 by Marcus and Moyls who showed they were (U, V)-operators with U and V nonsingular matrices. In 1966 Moore characterized the rank-2 preservers. For T nonsingular and rank-k preserver, Beasley and Djokovic independently characterized them.

In a series of articles beginning in 1970 and ending in 1992, Beasley showed that T preserves rank-k for any $k, 1 \leq k \leq m$ if and only if T is a (U, V)-operator where U and V are nonsingular matrices. The method used was to first establish a maximum dimension of a rank-k space, a space of matrices all of whose nonzero members have rank k, then show that if the mapping were singular that there should be a rank-k space of too large dimension, and thus get a contradiction implying that T must be nonsingular and appeal to previous results. In all these cases, T was shown to be a (U, V)-operator.

Let $\mathcal{N}$ denote the set of all nilpotent matrices in $\mathcal{M}_n(\mathbb{F})$ where $\mathbb{F}$ is a field with at least n elements and $n \geq 3$. Let T be a linear operator on $\mathcal{M}_n(\mathbb{F})$ which preserves $\mathcal{N}$. Clearly, if L is any linear functional on $\mathcal{M}_n(\mathbb{F})$ and N is a nilpotent matrix, then the operator $T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ such that $T(X) = L(X)N$ is a linear operator that preserves $\mathcal{N}$. However, if T is invertible, Botta, Pierce and Watkins in 1983 showed that T must be a (U, V)-operator with $U = V^{-1}$.

Let $\mathcal{I}$ be the set of all idempotent matrices and let $\mathcal{S}_n(\mathbb{F})$ denote the symmetric matrices in $\mathcal{M}_n(\mathbb{F})$. Let T be a linear operator on $\mathcal{M}_n(\mathbb{C}), \mathcal{M}_n(\mathbb{R}), \mathcal{S}_n(\mathbb{C})$, or $\mathcal{S}_n(\mathbb{R})$. In 1987, Chan, Lim and Tan showed that if $T(I) = I$ and T preserves $\mathcal{I}$ then T is a (U, V)-operator with $U = V^{-1}$.

5.3 Preservers of Relations on Matrices.

The most common relation on matrices are perhaps similarity, congruence, and equivalence. We say that A is similar to B if there is an invertible matrix S such that $A = S^{-1}BS$, A is congruent to B if there exits an invertible matrix S such that $A = S^*BS$ where S^* is the conjugate transpose of S, and A is equivalent to B if there exist invertible matrices S and T such that $A = SBT$. Preservers of similarity were characterized by Hiai in 1987 where it was shown that $T(A) = \text{tr}(A)B$ for all $A \in \mathcal{M}_n(\mathbb{C})$ where B is some fixed matrix; or, $T(A) = \alpha S^{-1}AS + \beta \text{tr}(A)I$ for all $A \in \mathcal{M}_n(\mathbb{F})$ or $T(A) = \alpha S^{-1}A^tS + \beta \text{tr}(A)I$ for all $A \in \mathcal{M}_n(\mathbb{F})$ where $\alpha, \beta \in \mathbb{C}$. If T preserves congruence, then Horn, Li and Tsing in 1991 showed that T is a (U, V)-operator where for some nonsingular matrix S, $U = \alpha S^*$ and $V = S$. If T preserves equivalence, then Horn, Li and Tsing in 1991 showed that T is a (U,V)-operator where U and V are nonsingular. (Here the matrices need not be square.)

Another common relation on matrices is that of commutativity. Preservers of this relation have been studied under many different settings, since $[AB - BA]$ is a commutator in an algebra. The set of commuting pairs is of interest to matrix theorists in general. Let $T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ be an *invertible* linear operator that preserves commuting pairs of matrices where F is an algebraically closed field. In 1976, Watkins for $n \geq 4$, and independently in 1978 Beasley for $n \geq 3$, showed that there is some $\alpha \in \mathbb{F}$, some linear functional $L : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ and an invertible matrix P

such that either $T(A) = L(A)I + \alpha P^{-1}AP$ for all $A \in \mathcal{M}_n(\mathbb{F})$ or $T(A) = L(A)I + \alpha P^{-1}A^tP$ for all $A \in \mathcal{M}_n(\mathbb{F})$. Beasley also showed that the condition that T be invertible could be replace by the condition: $AB = BA$ if and only if $T(A)T(B) = T(B)T(A)$. (In this case T is said to *strongly* preserve commutativity.)

6.0 The Inverse Preserver Problem.

This type of problem has not been successfully attacked. It asks the question: Given a linear transformation between two vector spaces, what sets are preserved.

An example. Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be the transformation such that $T(X) = UXV$ for all $X \in \mathcal{M}_n(\mathbb{S})$ where U and V are invertible and $\det(U)\det(V) \neq \pm 1$.

Then T preserves

- rank,
- sets of matrices of fixed rank,
- sets of matrices of boundfed rank,
- singular matrices,
- nonsingular matrices.

That is T preserves sets and functions defined by rank. Are there any others?

T does not preserve

- determinant,
- permanent,
- any other schur function,
- any symmetric function,
- commuting pairs,
- idempotence,
- nilpotence, etc.

Is this list complete?

Another Example

Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be the transformation such that $T(X) = PXP^t$ where P is a permutation matrix.

Then T preserves just about everything, except the row sum vector. That is $X\mathbf{j} \neq T(X)\mathbf{j}$ for all matrices X where $\mathbf{j}$ is the all ones vector. Is this all?