Linear Preserver Problems and their Solutions, Matrices over Semirings.

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Abstract In this article we will survey the research concerning linear transformations on "vector" spaces over semirings (which are not rings).

Introduction

A *semiring* is a set together with two binary operations we call addition and multiplication. The triple $(\mathbb{S}, +, \cdot)$ is a *semiring* if and only if for every $a, b, c \in \mathbb{S}$

1. $a + b \in \mathbb{S}$
2. $a \cdot b \in \mathbb{S}$
3. $a + (b + c) = (a + b) + c$
4. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
5. $a + b = b + a$
6. There exists $0 \in \mathbb{S}$ such that $a + 0 = 0 + a = a$ for all $a \in \mathbb{S}$

7. $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$.

The semiring is commutative if

8. $a \cdot b = b \cdot a$

and has a unity if

9. There exists $1 \in \mathbb{S}$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in \mathbb{S}$.

We say that $(\mathbb{S}, +, \cdot)$ is *antinegative* if

10. $a + b = 0$ implies $a = b = 0$.

We will suppress the operations and refer to $\mathbb{S}$ as the semiring and to indicate multiplication we use juxtaposition so that $ab = a \cdot b$.

Some common examples of semirings are $\mathbb{R}_+$, the semiring of all nonnegative real numbers; $\mathbb{Z}_+$, the set of nonnegative integers; E_+, the set of nonnegative even integers, etc.. For a set X we let 2^X denote the set of all subsets of X and in that case, 2^X is a semiring under the operations of union for addition and intersection as multiplication. The most common of these is the binary Boolean semiring, where $X = \{a\}$. Thus, $\mathbb{B} = \{0, 1\}$ is the binary Boolean semiring where all arithmetic is as for the integers except that $1 + 1 = 1$. We will drop the "binary" and just call $\{0, 1\}$ the Boolean semiring. All of these semirings are commutative and antinegative and all except E_+ have a unity.

Henceforth, all semirings will be commutative, antinegative and have a unity. Further they will have no zero divisors, that is, if $ab = 0$ then either $a = 0$ or $b = 0$.

A set $\mathcal{V}$ together with two binary operations, addition, $+$, and scalar multiplication, $\cdot$, is said to be a *semi-vector space over the semiring* $\mathbb{S}$ if and only if for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$, and $\alpha, \beta \in \mathbb{S}$,

1. $\mathbf{u} + \mathbf{v} \in \mathcal{V}$
2. $\alpha \cdot \mathbf{u} \in \mathcal{V}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
5. $\alpha \mathbf{v} = \mathbf{v} \alpha$
6. There exists $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = v$ for all $\mathbf{v} \in \mathcal{V}$

7. $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$

8. $1\mathbf{u} = \mathbf{u}$

9. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

10. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$

A semi-vector space has all the properties of a vector space over a field except that, as there are no negatives in $\mathbb{S}$, there are no negatives in a semi-vector space. Henceforth we will call a semi-vector space a vector space. A set of vectors in $\mathcal{V}$ is said to be linearly independent if no element in the set is a linear combination of the others. Other that this one difference, the concepts of basis , dimension, subspace, etc are parallel to the concepts in vector spaces over fields.

Let $T : \mathcal{V} \to \mathcal{V}$ be a mapping such that $T(\mathbf{0}) = \mathbf{0}$, and for $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, and $\alpha, \beta \in \mathbb{S}$, $T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v})$. Then T is called a *linear operator* on $\mathcal{V}$. The addition of the requirement that $T(\mathbf{0}) = \mathbf{0}$ is necessary due to the nature of antinegative semirings.

Let $\mathcal{M}_{m,n}(\mathbb{S})$ denote the set of all $m \times n$ matrices over $\mathbb{S}$. As in the field case, $\mathcal{M}_{m,n}(\mathbb{S})$ is a vector space over $\mathbb{S}$. Let $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$ be a linear operator. T is said to be a (U,V)-*operator* if either $T(A) = UAV$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$ or $m = n$ and $T(A) = UA^tV$ for all $A \in \mathcal{M}_{m,n}(\mathbb{S})$ where, unless otherwise noted, U and V are invertible matrices over $\mathbb{S}$ of appropriate orders.

In this article we will assume that $m \leq n$. Let I_n denote the $n \times n$ identity matrix, $J_{m,n}$ the $m \times n$ matrix of all ones and $O_{m,n}$ the $m \times n$ matrix of all zeros. Unless confusion is caused, we will suppress the subscripts and write I, J, O instead. Let $E_{i,j}$ denote the matrix whose (i, j) entry is one and all other entries are zero. The matrix $E_{i,j}$ is called a *cell*.

In this article we will be interested in what are known as "preserver problems". Let $\mathcal{K}$ be a subspace of $\mathcal{M}_{m,n}(\mathbb{S})$, usually $\mathcal{M}_{m,n}(\mathbb{S})$ itself. Let $T : \mathcal{K} \to \mathcal{K}$ be a linear operator. There are three basic types of preserver problems:

1. Given a function $f : \mathcal{K} \to \mathcal{S}$ determine the structure of all linear operators $T : \mathcal{K} \to \mathcal{K}$ such that $f(T(A)) = f(A)$ for all $A \in \mathcal{K}$. (Note Here $\mathcal{S}$ can be any set and f can be any type of function desired.)

2. Let $\mathcal{S}$ be a subset of $\mathcal{K}$. Determine the structure of all linear operators $T : \mathcal{K} \to \mathcal{K}$ such that $A \in \mathcal{S}$ implies that $T(A) \in \mathcal{S}$.

3. Let "$\sim$" be a relation of $\mathcal{K}$. Determine the structure of all linear operators $T : \mathcal{K} \to \mathcal{K}$ such that $A \sim B$ implies that $T(A) \sim T(B)$.

It may be noted that some preserver problems can be stated in more than one of the above types.

Let $A \in \mathcal{M}_{m,n}(\mathbb{S})$. Suppose that there are matrices $B \in \mathcal{M}_{m,k}(\mathbb{S})$ and $C \in \mathcal{M}_{k,n}(\mathbb{S})$ such that $A = BC$. Then, BC is called a k-factorization of A. Since $I_m A = A$ is an m-factorization, A always has factors. Let $A \neq O$ and let $r_{\mathbb{S}}(A)$ denote the smallest k for which A has a k-factorization.

Let $r_{\mathbb{S}}(O) = 0$. Then $r_{\mathbb{S}}(A)$ is called the *semiring rank* or *factor rank* of A. If $\mathbb{S} = \mathbb{B}$ we call the factor rank of A the *Boolean* rank of A.

Let $A, B \in \mathcal{M}_{m,n}(\mathbb{S})$ where $\mathbb{S}$ is antinegative. We say that A *dominates* B if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$, and we write $A \geq B$ or $B \leq A$. Also, if $A \geq$B we let $A \setminus B = c$ where $c_{i,j} = 0$ if and only if $a_{i,j} = 0$ or $b_{i,j} \neq 0$.

The Beginnings.

The beginnings of linear preserver theory on semirings was prompted by the work of several authors before 1980. They include Ki-Hang Kim (Butler), Lou Caccetta, Dominique deCaen, David Gregory, Norman Pullman to name a few. The book "Boolean Matrix Theory and Applications" by Kim and work of Norm Pullman, especially "A property of infinite products of Boolean matrices" laid the ground work for this study.

In 1967, Pullman observed without proof that the basis of a Boolean vector space is unique. For proof see Beasley and Pullman It is easily seen by the following example that a subspace of a Boolean vector space may have a larger dimension than the vector space itself.

Example 2.1 *Let $\mathcal{V} = \mathbb{B}^4$, the set of all $(0,1)$-4-tuples. Then $\mathcal{V}$ is a 4-dimensional vector space. Let $\mathcal{K} =$ Span($\mathcal{S}$) where*

$$\mathcal{S} = \{[1\ 1\ 0\ 0]^t, [1\ 0\ 1\ 0]^t, [1\ 0\ 0\ 1]^t, [0\ 1\ 1\ 0]^t,$$
$$[0\ 1\ 0\ 1]^t, [0\ 0\ 1\ 1]^t, [1\ 0\ 0\ 0]^t\}.$$

It is easily checked that $\mathcal{S}$ is linearly independent and, hence, is the basis for the subspace $\mathcal{K}$. Hence, $\dim(\mathcal{K}) = 7$.

The first article that appeared where a linear operator on a semi-vector space preserving a set or function was characterized was in 1984 when Beasley and Pullman characterized the factor rank preservers.

A subspace $\mathcal{K}$ of $\mathcal{M}_{m,n}(\mathbb{S})$ is called a rank rank-k space if every nonzero element of $\mathcal{K}$ has rank k. In this article, the following was shown:

Theorem 2.2 *Let* $\min\{m, n\} > 1$*. Let* $T : \mathcal{M}_n(B) \to \mathcal{M}_n(B)$ *be a linear operator. Then the following are equivalent:*

1. T *is invertible and preserves the set of rank-1 matrices.*

2. T *preserves the set of rank-1 matrices and* T *preserves the set of rank-2 matrices.*

3. T *preserves Boolean rank.*

4. T *is a (U,V)*$-operator$ *where* U *and* V *are invertible (hence permutation) matrices.*

Norman J. Pullman 1931-1999

The following was also proven about linear transformations on Boolean vector spaces in general:

Theorem 2.3 *Let $\mathcal{V}$ be a vector space over $\mathbb{B}$ and let T be a linear operator on $\mathcal{V}$. The following are equivalent:*

1. *T is invertible .*

2. *T is injective.*

3. *T is surjective.*

4. *T permutes the basis of $\mathcal{V}$*

5. *T preserves the dimension of every subspace of $\mathcal{V}$.*

During the next six years, before 1990, there were five more papers on linear preserver problems over semirings. They were all authored by Beasley and Pullman, with the addition of another coauthor, Gregory, on one. These results follow.

In 1985 Beasley, Gregory and Pullman showed essentially the same resutles as in the earlier paper but over subsemirings of $\mathbb{R}_+$, the semiring of nonnegative real numbers. This paper contained significant results concerning rank-k spaces over subsemirings of $\mathbb{R}_+$. They proved:

Theorem 2.4 *Let* $\min\{m,n\} \geq 4$*, let* $\mathbb{S}$ *be a subsemiring of* $\mathbb{R}_+$*, and let* $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$ *be a linear operator. Then the following are equivalent:*

1. T *is invertible and preserves the set of rank-1 matrices.*

2. T *preserves the set of rank-1 matrices,* T *preserves the set of rank-2 matrices and* T *preserves the set of rank-4 matrices.*

3. T *preserves factor rank.*

4. T *is a (U,V)*$-operator$ *where* U *and* V *are invertible.*

Let $\mathbb{S}$ be a linearly ordered set. That is, $\mathbb{S}$ is equivalent to a set of subsets of the real interval [0.1] such that each element is of the form $[0, \alpha]$ for some $0 \leq \alpha \leq 1$. Then, under the operations of union and intersection on the set of subsets of the real interval $[0, 1]$ such that each element is of the form $[0, \alpha]$ or max and min if considered as a subset of $[0, 1]$, $\mathbb{S}$ forms an antinegative semiring if $[0, 1] \in \mathbb{S}$ or $0 \in \mathbb{S}$ (resp.). If $[0, \sup_{[0,\alpha]\in\mathbb{S}}]$ is in $\mathbb{S}$, $\mathbb{S}$ is an antinegative with unity and no zero divisors. In this case we call $\mathbb{S}$ a *chain* semiring. If $\mathbb{S} = \{[0, \alpha] \mid \alpha \in [0, 1]\}$, $\mathbb{S}$ is called the *Fuzzy* numbers or the Fuzzy semiring.

In 1986, Beasley and Pullman proved Theorem 2.3 for operator on matrices over the Fuzzy semiring or indeed over any chain semiring.

Given any semiring, the permanent function is defined, unlike the determinant where negatives are required:

per (A) $= \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$ where S_n is the group af all permutations of the set $\{1, 2, \cdots, n\}$. Here, of course, the matrices are square.

Consider $A \in \mathcal{M}_{m,n}(\mathbb{S})$ where $\mathbb{S}$ is any antinegative semiring. The *term rank* of A is the least number of lines (rows or columns) that contain all the nonzero entries of A. The *rook polynomial* of A is
$R_A(x) = \sum_{j=1}^{\min\{m,n\}} p_j x^j$ where p_j is the sum of the permanents of the $j \times j$ submatrices of A. In 1987, Beasley and Pullman proved:

Theorem 2.5 *Let* $T : \mathcal{M}_{m,n}(\mathbb{S}) \to \mathcal{M}_{m,n}(\mathbb{S})$ *and* $\mathbb{S}$ *have only one unit. The following are equivalent:*

1. T *preserves the permanent (when* $m = n$ *).*

2. T *preserves the rook polynomial.*

3. T *preserves term rank.*

4. T *is a (U,V)*$-operator$ *where* U *and* V *are permutation matrices.*

This article also addressed more general semirings, those with not just one unit.

In 1988, Beasley and Pullman characterized the preservers of the r^{th} elementary symmetric (permanental) functions over antinegative semirings.

Let $T : \mathcal{V} \to \mathcal{V}$ be a linear operator, and let Ψ be a subset of $\mathcal{V}$. The operator T is said to *strongly preserve* Ψ if, $A \in \Psi$ if and only if $T(A) \in \Psi$.

Let $\mathbb{S}$ be any antinegative semiring. A matrix $A \in \mathcal{M}_n(\mathbb{S})$ is said to be *primitive* if A^k has all nonzero entries for some k. The minimal k such that A^k has all nonzero entries is called the *exponent* of A. When dealing with operator on matrices over , say $\mathbb{B}$, if T maps every element in $\mathcal{M}_n(\mathbb{B})$ to some fixed matrix $X \in \Psi$ then T preserves Ψ. One can investigate more interesting operators if one specifies additional conditions on T such as: T is injective; $T(J) = J$; or T strongly preserves Ψ.

These type of conditions usually restrict T to a class of operators that are more easily classified.

An operator needed in the next theorem is called a *diagonal replacement operator* which is an operator on $\mathcal{M}_n(\mathbb{B})$ such that $T(E_i, j) = E_{i,j}$ if $i \neq j$ and $T(E_{i,i} \leq I$ and $T(E_{i,i} \neq O$ for all $i = 1, \cdots, n$.

In 1989, Beasley and Pullman characterized linear operator that strongly preserve primitive matrices:

Theorem 2.6 *Let* $T : \mathcal{M}_n(\mathbb{B}) \rightarrow \mathcal{M}_n(\mathbb{B})$ *be a linear operator and* $n > 2$. *Then,* T *strongly preserves the set of primitive matrices if and only if T is one of, or a composition of some of, the following:*

- *transposition,*
- *(U,V)*$-operator$ *where* u *and* V *are permutation matrices and* $u = v^t$*, and,*
- *a nonsingular diagonal replacement operator.*

If $n = 2$ the following operator is also possible: $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b & (a+d) \\ c & 0 \end{bmatrix}$, for any $a, b, c, d \in \mathbb{B}$.

The technique of the proof of this theorem is used in many of the preserver theorems over semirings, and we sketch the proof here.

Suppose that T strongly preserves the set of all primitive matrices in $\mathcal{M}_n(\mathbb{B})$, and suppose that $T(X) = O$ for some $X \in \mathcal{M}_n(\mathbb{B})$. Then, $T(E_{i,j}) = O$ for some (i,j). Without loss of generality we may assume that $i = 1$ so that $T(E_{1,k}) = O$. Let $A = \sum_{i=1}^{n} E_{1,i} + \sum_{j=2}^{n} E_{j,1}$. Then A is primitive. so $T(A)$ must be primitive. However, if $B = A \setminus E_{1,k}$, then $T(B) = T(A)$, but B is not primitive thus $T(B)$ must not be primitive, a contradiction. Thus T is nonsingular.

Now, suppose that the image of some cell is not a cell, then we may construct a matrix A that has only n nonzero entries but whose image is primitive, again a contradiction. Thus T maps cells to cells. If the image of any two cells is the same cell it is possible to map a primitive matrix with $n+1$ nonzero entries to a matrix with n nonzero entries, which cannot be primitive, again a contradiction. Thus linear operators that strongly preserve primitivity preserves the set of cells.

To complete the proof, one shows that lines are mapped to lines and hence T permutes rows and T permutes columns. If the permutation that permutes the rows for T is not the same as the permutation that permutes the columns for T then one can construct a matrix which is primitive but whose image is not, a contradiction. Thus T is a (P, P^t)-operator.

To extend Theorem 2.6 to more general semirings, or indeed to any field, one observes that the concept of primitivity depends only upon the location of nonzero entries in the matrix, not on the magnitude.

Between 1990 and 2000, several researchers joined in the preserver problem quest over semirings. Those articles involving ranks of matrices over semirings were authored by Beasley

The Middle Years, 1990-2000.

During the 1990's the number of researcher of preserver problems over semirings increased to nearly 20 others. The first were coauthors, students and colleagues, of Beasley and Pullman, then later by their colleagues and finaly to others not academically related to those before. They include: B. R. Bapat, C-G.Cao, G.-S. Cheon, S.-H. Hong, S-G. Hwang, Y.-B. Jun, S.-G. Lee, G.-Y. Lee, Seon-J. Kim, Si-J. Kim, Steve Kirkland, S. Monson, S. Pati, S.-Z.Song, H.-K. Shin, S.-D. Yang and X. Zhang.

Most of the articles written were on some form of rank preservers over various semirings. The various forms of rank considered included column rank, max rank, spanning rank, term rank, zero term rank, and real rank over semirings those mentioned above and also over max algebras, etc. Most of the results were parallel to those researched earlier. S.-Z. Song and his collaborators have lead the way in these articles since the mid 1990's.

Other areas considered included commuting pairs , idempotent and r-potent matrices , the index of imprimitivity , Morse-Penrose inverses , upper ideals of matrices , and matrix properties defined by properties of bipartite and directed graphs, like chromatic number, small cycles , etc. .

Tye connection between graph theory and $(0, 1)$ matrix theory has been understood for some time. In 1984, Pullamn considered linear preservers of clique covering numbers. The clique covering number of a bipartite graph is the Boolean rank of the adjacency (reduced) matrix of that graph. Since then, linear preservers of subsets of graphs has been a active area of research.

Since the year 2000, there have been several publications each year about linear preservers over semirings. In the nex section we review a few of the recent results.

Recent Work.

Since the year 2000 there have been more than 30 more publications on linear preservers over semirings and the list of researchers in this endeavor has expanded by about 10 more. Some of those we specifically mention are listed below in our review of active areas.

We now review three areas where there has been a recent flurry of activity. They are perimeter preservers, inequality preservers pertaining to ranks, and tournament preservers. For emphasis we restate that all semirings we consider here are commutative, antinegative with unity and have no zero divisors.

Perimeter preservers.

Let $A \in \mathcal{M}_{m,n}(\mathbb{S})$ be factor rank 1. Then there are vectors, $\mathbf{u} \in \mathbb{S}^m$ and $\mathbf{v} \in \mathbb{S}^n$ such that $A = \mathbf{u}\mathbf{v}^t$. If the number of nonzero entries in $\mathbf{u}$ is u and the number of nonzero entries in $\mathbf{v}$ is v, then the *perimeter* of A, perim(A), is the sum $u + v$. Since $\mathbb{S}$ is antinegative and has no zero divisors, if $A = \mathbf{x}\mathbf{y}^t$, then the number of nonzero entries in $\mathbf{x}$ must also be u, etc. so the perimeter of a rank one matrix is well defined.

Let $A \in \mathcal{M}_{m,n}(\mathbb{S})$ and let $\mathbf{u}_i \in \mathbb{S}^m$ for $i = 1, \cdots, k$ and $\mathbf{v}_j \in \mathbb{S}^n$ for $j = 1, \cdots, k$ respectively, for . If $A = \sum_{i=1}^{n} \mathbf{u}_i \mathbf{v}_i^t$ is a rank one factorization of A, then this factorization is said to have perimeter the sum of the perimeters of the terms, ie., perim$(\sum_{i=1}^{n} \mathbf{u}_i \mathbf{v}_i^t) = \sum_{i=1}^{n}$ perim$(\mathbf{u}_i \mathbf{v}_i^t)$ The perimeter of A is the minimum of the perimeters of all rank one factorizations of A.

A type of (U,V)$-operator$ which we will be interested in in the next theorem is called a (P, Q, B)-operator. For $T : |m \rightarrow \mathcal{M}_{m,n}(\mathbb{S})$ T is a (P, Q, B)-operator if and only if there is some matrix $B \in \mathcal{M}_{m,n}(\mathbb{S})$ with $b_{i,j} \neq 0$ for all i, j, such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_{m,n}(\mathbb{S})$ or $m = n$ and $T(X) = P(X \circ B)^t Q$ for all $X \in \mathcal{M}_{m,n}(\mathbb{S})$ where P and Q are permutation matrices.

The first investigation into preservers of the perimeter of a matrix was by Kang and Song in 2003. since then there have been a few articles appear on this subject. The most recent was by Beasley, Kang and Song . This, the most general, proved:

Theorem 4.7 *Let* $T : \mathcal{M}_{m,n}(\mathbb{S}) \rightarrow \mathcal{M}_{m,n}(\mathbb{S})$ *and* $4 \leq k \leq mn - 2(n - m) + 4$*. If* T *preserves matrices of perimeter 2 and matrices of perimeter* k *then* T *is a* $P, Q, B)$*-operator. Further, if* $\mathbb{S}$ *has unique factorization and* $k \neq 5$ *then,* T *preserves matrices of perimeter 2 and matrices of perimeter* k *if and only if* T *is a* $P, Q, B)$*-operator with* $B = J$*.*

As in the case of primitive preservers, this theorem is first established in the Boolean case to get that T is a (U,V)$-operator$ and then show necessary modifications to get the final theorem.

Inequality Preservers.

Another area of recent interest has been that of operators that preserve matrix inequalities. Recall the rank-sum and rank product inequalities over the real or complex numbers:

- $\mid \text{rank}(A) - \text{rank}(B) \mid \leq \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)$;

- Sylvester's Laws:

$$\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB) \leq$$

$$\min\{\text{rank}(A), \text{rank}(B)\}$$

- the Frobenius inequality:

$$\text{rank}(AB) + \text{rank}(BC) \leq$$

$$\text{rank}(ABC) + \text{rank}(B),$$

where A, B, C are real or complex conformal matrices.

The following list of inequalities was proved by Beasley and Guterman.

If the semiring is arbitrary antinegative, then

1. $rank(A+B) \leq rank(A) + rank(B)$;

2. $rank(AB) \leq \min\{rank(A), rank(B)\}$.

   If the semiring uses Boolean arithmetic, then

3. $rank(A+B) \geq$

$$\begin{cases} rank(A) & \text{if} \quad B = O \\ rank(B) & \text{if} \quad A = O \\ 1 & \text{if} \quad A \neq O \text{ and } B \neq O \end{cases};$$

4. $rank(AB) \geq$

$$\begin{cases} 0 & if \quad rank(A) + rank(B) \leq n \\ 1 & if \quad rank(A) + rank(B) > n \end{cases}$$

The linear operators which preserve the extreme cases of these inequalities were characterized by Beasley, Guterman, Neal and Yi. An example is from Beasley and Guterman is:

Theorem 4.8 *Let $\mathcal{S}$ be an antinegative semiring and $T : \mathcal{M}_{m,n}(\mathcal{S}) \to \mathcal{M}_{m,n}(\mathcal{S})$ preserves $\mathcal{X} = \{(X, Y) \in \mathcal{M}_{m,n}(\mathbb{S})^2 \quad |$ $\text{rank}(X + Y) = \text{rank}(X) + \text{rank}(Y)\}$.*
If either

1. *T is surjective, or*

2. *$\mathbb{S}$ is finite or a chain semiring, and T strongly preserves $\mathcal{X}$*

then T is a (U,V)$-operator$ for $U = PD$ and $V = EQ$ where P and Q are permutation matrices and D and E are diagonal matrices of appropriate sizes.

Preservers of Tournament Matrices.

A recent endeavor in preserver problems over semiring is the preservation of various tournaments. A *tourmanent matrix* is a $(0,1)$-matrix, A, such that $a_{i,i} = 0$ for all i and $a_{i,j} = 1$ if and only if $a_{j,i} = 0$. That is, a $(0,1)$-matrix, A, is a tournament matrix if and only if $A + A^t = J - I$, where the addition is real. Let $\mathcal{M}_n(\mathbb{B})^{(0)}$ denote the st of all $A \in \mathcal{M}_n(\mathbb{B})$ such that the main diagonal of A is all zeros. In 2004, Beasley [**?**] proved:

Theorem 4.9 *Let* $T : \mathcal{M}_n(\mathbb{B})^{(0)} \to \mathcal{M}_n(\mathbb{B})^{(0)}$. *Then,* T *strongly preserves the set of all tournament matrices if and only if there exists a 1-1 mapping* $\sigma : \{(i,j) : 1 \leq i < j \leq n\} \to \{(i,j) : 1 \leq i,j \leq n\}$ *such that for* $i < j$ $T(E_{i,j}) = E_{\sigma(i,j)}$ *and* $T(E_{j,i}) = E^t_{\sigma(i,j)}$.

(Note the mapping σ is chosen such that the image under T of a tourmanent matrix is a tournament matrix).

Linear operator on tournaments was continued by Beasley, Brown and Guterman in [**?**]. An interesting outcome of that paper is the next theorem, due to Guterman.

Remark 4.10 *Next we define a special type of transformation which we will use in the sequel. This type of transformation shows that the class of operators which preserve term rank-1 on $\mathcal{M}_n(B)^{(0)}$ is more diverse than it might be expected.*

Let $C_i^{(0)} = \sum\limits_{l \neq i} E_{l,i}$, $R_j^{(0)} = \sum\limits_{l \neq j} E_{j,l}$.

Definition 4.11 *For the fixed indices i, p, s, r, u, $i \neq p$, $r \neq u$, $s \neq r$ and $s \neq u$, a linear transformation $\Xi_{i,p,s,r,u} : \mathcal{M}_n(B)^{(0)} \to \mathcal{M}_n(B)^{(0)}$ is defined as follows:*

- *for any* a, b *such that* $a \notin \{i, p\}$, $b \notin \{a, i, p\}$ *the image* $\Xi_{i,p,s,r,u}(E_{a,b})$ *is an arbitrary nonzero matrix dominated by* $C_s^{(0)}$,

- $\Xi_{i,p,s,r,u}(E_{i,p})$ *is an arbitrary nonzero matrix dominated by* $R_r^{(0)}$,

- $\Xi_{i,p,s,r,u}(E_{p,i})$ *is an arbitrary nonzero matrix dominated by* $R_u^{(0)}$,

- $\Xi_{i,p,s,r,u}(E_{i,x}) = \Xi_{i,p,s,r,u}(E_{x,p}) = E_{r,s}$, *for any* $x \notin \{i, p\}$,

- $\Xi_{i,p,s,r,u}(E_{x,i}) = \Xi_{i,p,s,r,u}(E_{p,x}) = E_{u,s}$ *for any* $x \notin \{i, p\}$.

On the other matrices $\Xi_{i,p,s,r,u}$ *is defined by linearity.*

Theorem 4.12 *Let $T : \mathcal{M}_n(B)^{(0)} \to \mathcal{M}_n(B)^{(0)}$ be an additive operator, $n > 3$, $T(0) = 0$. Then, T preserves term rank-1 if and only if T is one of the operators listed below or a composition of some of them*

1. *a (U, V)-operator, where U has no more than one non-zero element in each column and V has no more than one non-zero element in each row;*

2. *there exist indices i, p, s, r, u such that $T(X) = \Xi_{i,p,s,r,u}(X)$ for all $X \in \mathcal{M}_n(B)^{(0)}$ or $T(X) = (\Xi_{i,p,s,r,u}(X))^t$ for all $X \in \mathcal{M}_n(B)^{(0)}$;*

3. *for some i, $T(X) \leq R_i^{(0)}$ for all $X \in \mathcal{M}_n(B)^{(0)}$;*

4. *for some j, $T(X) \leq C_j^{(0)}$ for all $X \in \mathcal{M}_n(B)^{(0)}$;*

5. *there exist indices* i, j, k, l *with* $i \neq j$ *and* $k \neq l$ *such that* $T(R_i^{(0)}) \leq R_k^{(0)}$, $T(C_j^{(0)}) \leq C_l^{(0)}$ *and* $T(E_{x,y}) = E_{k,l}$ *for all* $x \neq i$ *and* $y \neq j$*; or*

6. *there exist indices* i, j, k, l *with* $i \neq j$ *and* $k \neq l$ *such that* $T(R_i^{(0)}) \leq C_l^{(0)}$, $T(C_j^{(0)}) \leq R_k^{(0)}$ *and* $T(E_{x,y}) = E_{k,l}$ *for all* $x \neq i$ *and* $y \neq j$.