Preserving Regular Tournaments and Term Rank-1*

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Abstract

We investigate linear operators which map certain types of tournaments to themselves. To this end we also characterize term rank-1 preservers on the set of matrices whose associated digraphs are simple loopless directed graphs, and find that this set of operators is more diverse than might be expected.

Preliminaries

Definition 1.1 *A* binary Boolean *semiring,* $\mathbf{B}$*, is the set* $\{0,1\}$ *with the operations:*

$$\begin{array}{cc} 0+0=0 & 0\cdot 0=0 \\ 0+1=1+0=1 & 0\cdot 1=1\cdot 0=0 \\ 1+1=1 & 1\cdot 1=1. \end{array} \tag{1}$$

We will not use the term "binary" in the sequel since in this paper we consider only binary Boolean semirings. We will say that the arithmetic on the set $\{0,1\}$ is *Boolean* if it satisfies the conditions (1).

The union of two digraphs on the same set of vertices corresponds to the sum of the two adjacency matrices if the arithmetic used is Boolean.

Let $\mathcal{M}_n(\mathbf{B})$ denote the set of all $n \times n$ Boolean matrices. Let $\mathcal{M}_n(\mathbf{B})^{(0)}$ be the set of all $n \times n$ Boolean matrices with zero diagonal. We let $E_{i,j}$ denote the matrix with a "1" in the (i,j) entry and zero elsewhere. We call such a matrix a *cell*. A *line* of a matrix is a row or column of that matrix, and a *line matrix* is a matrix that has all its nonzero entries in one line. We denote $C_i^{(0)} = \sum\limits_{l \neq i} E_{l,i}$, $R_j^{(0)} = \sum\limits_{l \neq j} E_{j,l}$. The matrix I_n is the $n \times n$ identity matrix, J_n is the $n \times n$ matrix of all ones, O_n is the $n \times n$ zero matrix. We omit the subscripts when the order is obvious from the context and we write I, J, and O, respectively. We let $|A|$ denote the number of nonzero entries in the matrix A. For $\alpha = (i_1, \ldots, i_r)$, $\beta = (j_1, \ldots, j_s)$, we denote by $A[\alpha|\beta]$ the $r \times s$-submatrix of A which lies in the intersection of the $i_1, \ldots, i_r$-th rows and $j_1, \ldots, j_s$-th columns.

Definition 1.2 *A matrix A is said to* dominate *or* majorize *the matrix B, written $A \geq B$ or $B \leq A$, if $b_{i,j} \neq 0$ implies $a_{i,j} \neq 0$.*

Definition 1.3 *If A and B are matrices and $A \geq B$ we let $A \backslash B$ denote the matrix C where*

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0 \\ a_{i,j} & \text{otherwise} \end{cases} .$$

We denote $K := J \setminus I$.

Definition 1.4 *A nonempty set $\mathcal{V}$ is a* Boolean semimodule *if it is closed under addition, $\mathbf{0} \in V$, $\mathbf{v} + \mathbf{v} = \mathbf{v}$, $1 \cdot \mathbf{v} = \mathbf{v} \cdot 1 = \mathbf{v}$, and $0 \cdot \mathbf{v} = \mathbf{v} \cdot 0 = \mathbf{0}$ for all $\mathbf{v} \in V$.*

Thus $\mathcal{M}_n(\mathbf{B})$ and $\mathcal{M}_n(\mathbf{B})^{(0)}$ are Boolean semimodules.

Definition 1.5 *We say a mapping T between two Boolean semimodules is* linear *if* $T(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha T(\mathbf{x})+\beta T(\mathbf{y})$ *for any two elements* $\mathbf{x}, \mathbf{y}$ *of the semimodule and any* $\alpha, \beta \in \mathbf{B}$*.*

Remark 1.6 *Since the only scalars in* $\mathbb{B}$ *are* 0 *and* 1*, the linearity of* T *is equivalent to the following:* T *is additive and* $T(\mathbf{0}) = \mathbf{0}$*.*

Definition 1.7 *A linear operator is said to be* nonsingular *if* $T(\mathbf{x}) = \mathbf{0}$ *if and only if* $\mathbf{x} = \mathbf{0}$*.*

Definition 1.8 *We say that an operator,* $T : \mathcal{M}_n(\mathbf{B}) \to \mathcal{M}_n(\mathbf{B})$*,* preserves (strongly preserves) *a set* $\mathcal{X}$ *if for a matrix* $A \in \mathcal{X}$ *the matrix* $T(A)$ *is also in* $\mathcal{X}$ *(*$A \in \mathcal{X}$ *if and only if* $T(A) \in \mathcal{X}$*).*

Let us introduce the following two types of operators on $\mathcal{M}_{m,n}(\mathbf{B})$ which will be useful in the subsequent discussion.

Definition 1.9 *An operator* $T : \mathcal{M}_{m,n}(\mathbf{B}) \to \mathcal{M}_{m,n}(\mathbf{B})$ *is called a* (U, V)-operator *if there exist matrices* U *and* V *of appropriate orders such that* $T(X) = UXV$ *for all* $X \in \mathcal{M}_{m,n}(\mathbf{B})$*, or, if* $m = n$*,* $T(X) = UX^tV$ *for all* $X \in \mathcal{M}_{m,n}(\mathbf{B})$*, where* X^t *denotes the transpose of* X*.*

Definition 1.10 *An operator* T *is called a* (P, Q)-operator *if there exist permutation matrices* P *and* Q *such that* $T(X) = PXQ$ *for all* $X \in \mathcal{M}_n(\mathbf{B})$*, or* $T(X) = PX^tQ$ *for all* $X \in \mathcal{M}_n(\mathbf{B})$*.*

Term Rank-1 Preservers

Recall that $\mathcal{M}_n(\mathbf{B})^{(0)}$ is the set of all $n \times n$-matrices with zero diagonal, $K := J \setminus I$, $C_i^{(0)} = \sum_{l \neq i} E_{l,i}$, $R_j^{(0)} = \sum_{l \neq j} E_{j,l}$.

Definition 2.11 *The matrix $A \in \mathcal{M}_n(\mathbf{B})$ is said to be of* term-rank k *($t(A) = k$) if the least number of lines (rows or columns) needed to include all nonzero elements of A is equal to k. The term-rank of a digraph is the term-rank of the adjacency matrix of this digraph.*

Lemma 2.12 *Let $\mathcal{K} \subseteq \mathcal{M}_n(\mathbf{B})$ be a semi-module, $T : \mathcal{K} \to \mathcal{K}$ be a surjective additive operator. Then $T(O) = O$ and T is linear.*

Proof. Since T is surjective there is some X with $T(X) = O$. Since $O \leq X$, $T(O) \leq T(X) = O$. Thus $T(O) = O$.

By Remark 1.6, T is linear. ∎

Lemma 2.13 *Let* $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ *be an additive operator with* $T(O) = O$ *that preserves term rank-1. Then* $T(A) \neq O$ *for all nonzero* $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$*.*

Proof. Let A be a nonzero matrix in $\mathcal{M}_n(\mathbf{B})^{(0)}$, then $A = \sum_{i \neq j} a_{i,j} E_{i,j}$ for some $a_{i,j} \in \mathbf{B}$. Since T preserves term rank-1, $T(E_{i,j})$ is nonzero for each i and j. The result now follows by the linearity of T. ∎

Remark 2.14 *Next we define a special type of transformation which we will use in the sequel. This type of transformation shows that the class of operators which preserve term rank-1 on* $\mathcal{M}_n(\mathbf{B})^{(0)}$ *is more diverse than it might be expected.*

Definition 2.15 *For the fixed indices* i, p, s, r, u, $i \neq p$, $r \neq u$, $s \neq r$ *and* $s \neq u$, *a linear transformation* $\Xi_{i,p,s,r,u} : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ *is defined as follows:*

- *for any* a, b *such that* $a \notin \{i, p\}$, $b \notin \{a, i, p\}$ *the image* $\Xi_{i,p,s,r,u}(E_{a,b})$ *is an arbitrary nonzero matrix dominated by* $C_s^{(0)}$,

- $\Xi_{i,p,s,r,u}(E_{i,p})$ *is an arbitrary nonzero matrix dominated by* $R_r^{(0)}$,

- $\Xi_{i,p,s,r,u}(E_{p,i})$ *is an arbitrary nonzero matrix dominated by* $R_u^{(0)}$,

- $\Xi_{i,p,s,r,u}(E_{i,x}) = \Xi_{i,p,s,r,u}(E_{x,p}) = E_{r,s}$, *for any* $x \notin \{i, p\}$,

- $\Xi_{i,p,s,r,u}(E_{x,i}) = \Xi_{i,p,s,r,u}(E_{p,x}) = E_{u,s}$ *for any* $x \notin \{i, p\}$.

On the other matrices $\Xi_{i,p,s,r,u}$ is defined by linearity.

Namely, for fixed i, p, s, r, u with $i \neq p$ we fix a column, $C_s^{(0)}$, and two different rows, $R_r^{(0)}$ and $R_u^{(0)}$ in the image. Then $\Xi_{i,p,s,r,u}$ maps all cells which are not in the i'th and p'th rows and not in the i'th and p'th columns to the chosen column $C_s^{(0)}$ in an arbitrary but nonzero way. The matrix $E_{p,i}$ is mapped to the chosen row $R_u^{(0)}$ in an arbitrary nonzero way, and the matrix $E_{i,p}$ is mapped to $R_r^{(0)}$ in an arbitrary nonzero way.

The other cells from the i'th row and p'th column are mapped to $E_{r,s}$, and the other cells from the i'th column and p'th row are mapped to $E_{u,s}$. We write Ξ without indices if this does not lead to any misunderstanding.

We note that $\Xi_{i,p,s,r,u}$ preserves term rank-1, since the image of any term rank one matrix dominated by row $x \neq i, p$ is dominated by column s; the image of a term rank one matrix dominated by row i is dominated by row r; and the image of any term rank one matrix dominated by row p is dominated by row u. In each case the term rank of the image is one. Similarly any term rank one matrix dominated by a column has term rank one.

Note that many of the transformations from Definition 2.15 are non-standard, i.e., they can not be extended to a transformation on the set of all matrices in such a way that it becomes either a (U, V)-operator where U has no more than one non-zero entry in each column and V has no more than one non-zero entry in each row, or a (P, Q)-operator. The reason is that both these (U, V) and (P, Q)-operators preserve term rank-1 on $\mathcal{M}_n(\mathbf{B})$.

However, in spite of the fact that it can be easily seen that Ξ preserves term rank-1 on $\mathcal{M}_n(\mathbf{B})^{(0)}$, no extension of the transformation defined on the whole matrix algebra, $\mathcal{M}_n(\mathbf{B})$, preserves term rank-1.

Let us consider some examples of such transformations.

Example 2.16 *Suppose* $n = 3$. *Let us consider the transformation* $\Xi_{1,2,2,1,3}$ *defined by:*

$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & b+f & a \\ 0 & 0 & 0 \\ c & d+e & 0 \end{bmatrix}.$$

For this operator, there is no way to assign the image of $E_{1,1}$ *to extend this operator to one on* $\mathcal{M}_3(\mathbf{B})$ *that preserves term rank-1, since if there were, the image of* $E_{1,1}$ *would have to be in both row one and row three.*

Now, suppose $n = 4$. *We consider* $\Xi_{1,4,2,1,3}$ *defined as follows:*

$$\begin{bmatrix} 0 & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & 0 & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & 0 & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 0 & a_{1,2} + a_{1,3} + a_{2,3} + a_{3,4} + a_{2,4} & 0 & a_{1,4} \\ 0 & 0 & 0 & 0 \\ 0 & a_{2,1} + a_{3,1} + a_{3,2} + a_{4,3} + a_{4,2} & 0 & a_{4,1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, as in the example in case $n = 3$, *considering the possible images of* $E_{1,1}$, *it is easily seen that no extension of this transformation will preserve term rank-1 on* $\mathcal{M}_4(\mathbf{B})$ *since the image of* $E_{1,1}$ *would have to be in the intersection of rows 1 and 3.*

In general, if we define $\Xi_{1,n,2,1,3}$ *such that* $E_{1,n} \to E_{1,n}, E_{n,1} \to E_{3,n}, E_{1,x} \to E_{1,2}$ *for* $2 \leq x \leq n-1$*,* $E_{x,n} \to E_{1,2}$ *for* $2 \leq x \leq n-1$ *and* $E_{u,v} \to E_{3,2}$ *for* $2 \leq u \leq n, 1 \leq v \leq n-1, u \neq v$*, and* $(u,v) \neq (n,1)$*, then no assignment of the image of* $E_{1,1}$ *will produce a linear operator on* $\mathcal{M}_n(\mathbf{B})$ *that preserves term rank-1, since the image of* $E_{1,1}$ *would have to be in the intersection of rows 1 and 3.*

Lemma 2.17 *Let* $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ *be an additive operator,* $n > 3$*,* $T(O) = O$*. If* T *preserves term rank-1 then one of the following possibilities hold:*

(i) T *maps all rows to rows and all columns to columns;*

(ii) T *maps all rows to columns and all columns to rows;*

(iii) there exist indices i, p, s, r, u *such that* $T(X) = \Xi_{i,p,s,r,u}(X)$ *for all* $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$ *or* $T(X) = (\Xi_{i,p,s,r,u}(X))^t$ *for all* $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$*;*

(iv) there exist indices r, s *such that* $T(K) \leq C_s^{(0)} + R_r^{(0)}$*.*

Proof. Let T be an additive term rank-1 preserver. Then T sends each line of a matrix into a line, since otherwise the image of a term rank-1 matrix has the term rank grater than or equal to 2. Let us assume now that the cases (i), (ii), (iv) of Lemma 2.17 are not satisfied. Note that in this case $n \geq 3$.

Up to a permutations of rows and columns, we may assume that $T(R_1^{(0)}) \leq R_1^{(0)}$, $T(R_2^{(0)}) \leq C_s^{(0)}$, i.e., the 1-st row is transformed to the 1-st row, and the 2-nd row is transformed to the s-th column. Thus for any $j \geq 3$ we have that $T(E_{1,j}) \leq R_1^{(0)}$ and $T(E_{2,j}) \leq C_s^{(0)}$. Hence, either $T(C_j^{(0)}) \leq R_1^{(0)}$ or $T(C_j^{(0)}) \leq C_s^{(0)}$.

We consider the following 4 cases now:

Case 1. At least two of the columns $C_3^{(0)}, \dots, C_n^{(0)}$ are transformed to $R_1^{(0)}$ and at least two of them are transformed to $C_s^{(0)}$. Without loss of generality we assume that $T(C_3^{(0)}) \leq R_1^{(0)}$, $T(C_4^{(0)}) \leq R_1^{(0)}$ and $T(C_5^{(0)}) \leq C_s^{(0)}$,

$T(C_6^{(0)}) \le C_s^{(0)}$. We are going to show that case (iv) holds. Indeed, it remains to check that $C_1^{(0)}$ and $C_2^{(0)}$ are majorized by $R_1^{(0)} + C_s^{(0)}$. Let us show that for any $j > 2$, $T(E_{j,1}) \le R_1^{(0)} + C_s^{(0)}$. For any $j \ge 3$ we have that either $j \ne 3$ or $j \ne 4$. Without loss of generality we assume that $j \ne 3$. Let us denote $T(E_{2,1}) = E_{r,s}$, $T(E_{j,3}) = E_{1,k}$. Hence $T(E_{j,1}) \le R_r^{(0)} + C_s^{(0)}$ and $T(E_{j,1}) \le R_1^{(0)} + C_k^{(0)}$, which implies $T(E_{j,1}) \le R_1^{(0)} + C_s^{(0)}$. Similar considerations with $E_{l,2}, E_{1,2}$, and $E_{l,5}$ or $E_{l,6}$ show that $T(E_{l,2}) \le R_1^{(0)} + C_s^{(0)}$ for any $l \ge 3$, i.e., any cell is dominated by the 1-st row and s-th column.

Case 2. The transformation T maps all except one of the columns $C_3^{(0)}, \dots, C_n^{(0)}$ to $R_1^{(0)}$ and the remaining column to $C_s^{(0)}$. Without loss of generality we assume that $T(C_n^{(0)}) \le C_s^{(0)}$. Similar to Case 1 we get that $T(E_{i,j}) \leq R_1^{(0)} + C_s^{(0)}$ for all $i = 1, \ldots, n-1$, $j = 1, \ldots, n$ and either $T(R_n^{(0)}) \leq R_1^{(0)}$, which leads to (iv), or $T(R_n^{(0)}) \leq C_t^{(0)}$ for some t. In the last case if $t = s$, then we obtain (iv) again, so it remains to consider the case $s \neq t$.

Since $T(R_2^{(0)}) \leq C_s^{(0)}$ and $T(C_j^{(0)}) \leq R_1^{(0)}$ for all $3 \leq j < n$, it follows that $T(E_{2,j}) \leq E_{1,s}$ for $3 \leq j \leq n-1$.

Now since $E_{2,1}$ and $E_{n,1}$ are both from the 1-st column we get from $T(E_{2,1}) \leq C_s^{(0)}$, $T(E_{n,1}) \leq C_t^{(0)}$, and $s \neq t$ that there is some k such that $T(E_{2,1}) + T(E_{n,1}) \leq R_k^{(0)}$. If there is some $E_{i,j}$, $i, j \geq 3$, such that $T(E_{i,j}) \leq R_1^{(0)}$ and $T(E_{i,j}) \not\leq E_{1,s}$ then it is easy to see, that $k = 1$. Otherwise, it follows that $T(E_{i,j}) \leq C_s^{(0)}$ for all i and $j > 1$, which implies that $T(E_{i,j})$ are majorized by $R_k^{(0)}$ or $C_s^{(0)}$, i.e., the case (iv).

So, we get that $T(E_{2,1}) = E_{1,s}$ and $T(C_1^{(0)}) \leq R_1^{(0)}$, and we have proved that the images of all cells, except for the cells from 2-nd column, are dominated by $R_1^{(0)} + C_s^{(0)}$.

Let us consider the image of $C_2^{(0)}$ now. If it is a row, then it is dominated by $R_1^{(0)}$, and we are in the case (iv) again. Otherwise $T(C_2^{(0)}) \leq C_t^{(0)}$. Thus $T(R_i^{(0)}) \leq R_1^{(0)}$ for all $i = 3, \ldots, n-1$, and therefore $T(X) = (\Xi_{2,n,1,s,t}(X))^t$, so we get the case (iii).

Case 3. The transformation T maps all except one of the columns $C_3^{(0)}, \ldots, C_n^{(0)}$ to $C_s^{(0)}$ and the remaining column to $R_1^{(0)}$. By applying the transposition operator we get Case 2.

Case 4. The transformation T maps all of the columns $C_3^{(0)}, \ldots, C_n^{(0)}$ to $R_1^{(0)}$ or all columns to $C_s^{(0)}$.

Without loss of generality we consider only the case, when T maps all of the columns $C_3^{(0)}, \dots, C_n^{(0)}$ to $R_1^{(0)}$. If T transforms $C_2^{(0)}$ to a row, then it is necessarily $R_1^{(0)}$. Then $C_1^{(0)}$ is mapped to $R_1^{(0)} + C_s^{(0)}$ as well, and case (iv) holds.

Thus we may assume that $T(C_2^{(0)})$ is majorized by some column, say $C_t^{(0)}$.

If $t = s$, we are in the case (iv) again. So, assume that $s \neq t$. Then $T(R_i^{(0)}) \leq R_1^{(0)} + C_t^{(0)}$ for all $i = 3, \dots, n$, and there is a row which is not mapped to $R_1^{(0)}$. It follows now that $T(C_1^{(0)})$ is a row, and after a transposition we are in conditions of Cases 1 or 2, which concludes the proof. ▪

Theorem 2.18 *Let $T : \mathcal{M}_n(\mathrm{B})^{(0)} \to \mathcal{M}_n(\mathrm{B})^{(0)}$ be an additive operator, $n > 3$, $T(0) = 0$. Then, T preserves term rank-1 if and only* *if T is one of the operators listed below or a composition of some of them*

1. *a (U, V)-operator, where U has no more than one non-zero element in each column and V has no more than one non-zero element in each row;*

2. *there exist indices i, p, s, r, u such that $T(X) = \Xi_{i,p,s,r,u}(X)$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$ or $T(X) = (\Xi_{i,p,s,r,u}(X))^t$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$;*

3. *for some i, $T(X) \leq R_i^{(0)}$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$;*

4. *for some j, $T(X) \leq C_j^{(0)}$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$;*

5. *there exist indices* i, j, k, l *with* $i \neq j$ *and* $k \neq l$ *such that* $T(R_i^{(0)}) \leq R_k^{(0)}$, $T(C_j^{(0)}) \leq C_l^{(0)}$ *and* $T(E_{x,y}) = E_{k,l}$ *for all* $x \neq i$ *and* $y \neq j$*; or*

6. *there exist indices* i, j, k, l *with* $i \neq j$ *and* $k \neq l$ *such that* $T(R_i^{(0)}) \leq C_l^{(0)}$, $T(C_j^{(0)}) \leq R_k^{(0)}$ *and* $T(E_{x,y}) = E_{k,l}$ *for all* $x \neq i$ *and* $y \neq j$.

Proof. Note that $T(X) = O$ if and only if $X = O$ since T preserves term rank-1.

We first show that if (i), or (ii) in Lemma 2.17 holds then T is (U, V)-operator.

Indeed, up to a transposition transformation we may assume that (i) holds. Since rows are transformed to rows and columns to columns, T induces the actions on the sets of rows and columns, correspondingly.

Let us denote by π_r, π_c the mappings on the set $\{1, \ldots, n\}$ such that $T(R_i^{(0)}) \le R_{\pi_r(i)}^{(0)}$ and $T(C_j^{(0)}) \le C_{\pi_c(j)}^{(0)}$, correspondingly. Since each cell is dominated by exactly one row and exactly one column it follows that $T(E_{i,j}) = E_{\pi_r(i),\pi_c(j)}$. Thus there exist matrices U and V with $u_{i,j} = 1$ iff $\pi_r(i) = j$ and $v_{i,j} = 1$ iff $\pi_c(j) = i$ such that $T(E_{i,j}) = UE_{i,j}V$. It follows from the additivity of T that $T(X) = UXV$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$, i.e., T is a (U, V)-operator.

Now, U has no more than one non-zero entry in each column and V has no more than one non-zero entry in each row, since any cell is mapped to a cell by above.

If (iii) holds in Lemma 2.17 then (2) holds in this theorem. Assume that (iv) holds in Lemma 2.17. Then there are three possibilities: for some i, $T(X) \le R_k^{(0)}$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$; for some j, $T(X) \le C_l^{(0)}$ for all

$X \in \mathcal{M}_n(\mathbf{B})^{(0)}$; or T maps some cells into row k not column l and some cells into column l not row k and $T(X) \leq R_k^{(0)} + C_l^{(0)}$ for all $X \in \mathcal{M}_n(\mathbf{B})^{(0)}$. Suppose that $T(E_{i,s}) = E_{k,r}$ and $T(E_{t,j}) = E_{q,l}$ where $r \neq l$ and $q \neq k$. Now, both $E_{i,j} + E_{i,s}$ and $E_{i,j} + E_{t,j}$ are term rank-1, thus $T(E_{i,j})$ is in the intersection of row k and column l, that is $T(E_{i,j}) = E_{k,l}$. Then, $T(R_i^{(0)}) \leq R_k^{(0)}$ and $T(C_j^{(0)}) \leq C_l^{(0)}$. Now consider $T(E_{x,y})$ with $x \neq i$ and $y \neq j$. Since $E_{i,y} + E_{x,y}$ and $E_{x,j} + E_{x,y}$ are term rank-1, $T(E_{x,y})$ must lie in the intersection of row k and column l, that is $T(E_{x,y}) = E_{k,l}$. That is (5) holds. The case $T(R_i^{(0)}) \leq C_l^{(0)}$ and $T(C_j^{(0)}) \leq R_k^{(0)}$ yields (6).

Conversely, each of the operators, and hence compositions of them, preserve term rank-1. ∎

Let us consider the cases $n = 2, 3$ separately.

Remark 2.19 *Note that the fact that Lemma 2.17 is true in case $n = 2$ is trivial by examining the few possible cases.*

Recall that $K = J \setminus I$.

Lemma 2.20 *Let $T : \mathcal{M}_3(\mathbf{B})^{(0)} \to \mathcal{M}_3(\mathbf{B})^{(0)}$ be an additive operator and $T(O) = O$. If T preserves term rank-1 then either T is a (P, P^t)-operator, where P is a permutation matrix or $T(A)$ has a zero row or a zero column for all $A \in \mathcal{M}_3(\mathbf{B})^{(0)}$.*

Proof. Suppose $T(K)$ has no zero row or column. We first show that T maps cells to cells. Suppose not, then without loss of generality we may assume that $T(E_{1,2}) = E_{1,2} + E_{1,3}$. Then, since $E_{1,2} + E_{1,3}$ and $E_{1,2} + E_{3,2}$ have term rank-1, $T(E_{1,3}) \leq R_1^{(0)}$ and $T(E_{3,2}) \leq R_1^{(0)}$. Since $E_{1,3} + E_{2,3}$ has term rank-1, $T(E_{2,3})$ cannot have a nonzero entry in column 1.

Since $T(E_{3,2}) \leq R_1^{(0)}$, $T(E_{3,1})$ cannot have a nonzero entry in column 1. Thus $T(E_{2,1})$ must have a nonzero entry in column 1.

If $T(E_{2,1}) = E_{2,1} + E_{3,1}$ then we must have $T(E_{2,3}) \leq C_1^{(0)}$, a contradiction.

If $T(E_{2,1}) = E_{2,1} + E_{2,3}$ then we must have that $T(E_{2,3}) = E_{2,3}$ and $T(E_{3,1}) \leq R_1^{(0)}$, so that $T(K)$ has no nonzero entry in row 3, a contradiction.

If $T(E_{2,1}) = E_{2,1}$, we must have that $T(E_{2,3}) = E_{2,3}$. Thus $T(E_{3,1}) = E_{2,3}$, but then $T(K)$ has no nonzero entry in the third row.

If $T(E_{2,1}) = E_{3,1}$ we get, by similar arguments, that $T(K)$ has no nonzero entry in the second row.

Thus T maps cells to cells.

Suppose that T is not one-to-one on the set of cells, then there are two cases: T maps two collinear cells to the same cell or T maps two noncollinear cells to the same cell.

Suppose that T maps two collinear cells to the same cell. Without loss of generality we may assume that $T(E_{1,2}+E_{1,3}) = E_{1,2}$. Here there are three possible choices: $T(E_{2,3}) = E_{1,2}, E_{1,3}$, or $E_{3,2}$.

If $T(E_{2,3}) = E_{1,2}$ then $T(E_{21}) \leq R_1^{(0)} + C_2^{(0)}$ and $T(E_{3,2}) \leq R_1^{(0)} + C_2^{(0)}$. That is, $T(E_{1,2}+E_{1,3}+E_{2,1}+E_{2,3}+E_{3,2}) \leq R_1^{(0)} + C_2^{(0)}$. It follows that $T(E_{3,1}) \leq R_2^{(0)} \cap C_1^{(0)}$ or else $T(K)$ has a zero row or column. That is $T(E_{3,1}) = E_{2,1}$. It now follows that $T(E_{3,2})$ is a cell in the intersection of the two sets $\{E_{1,2}, E_{1,3}, E_{3,2}\}$ and $\{E_{2,1}, E_{2,3}, E_{3,1}\}$, which is empty, a contradiction.

If $T(E_{2,3}) = E_{1,3}$ then $T(E_{3,2}) \leq R_1^{(0)} + C_3^{(0)}$ and $T(E_{2,1}) \leq R_1^{(0)} + C_3^{(0)}$. In this case we must have that $T(E_{3,1}) = E_{3,1}$ or else $T(K)$ would have a zero row or column. But then $T(E_{2,1} + E_{3,1})$ must have term rank 2, a contradiction.

If $T(E_{2,3}) = E_{3,2}$ then $T(E_{2,1}) \leq R_3^{(0)} + C_2^{(0)}$ and $T(E_{3,2}) \leq R_1^{(0)} + C_2^{(0)}$. In this case we must have that $T(E_{3,1}) \leq R_2^{(0)}$ or else $T(K)$ would have a zero second row. Since $T(E_{2,1}+E_{3,1})$ must have term rank-1, it follows that $T(E_{2,1}) = E_{3,1}$ and $T(E_{3,1}) = E_{2,1}$. But then $T(E_{3,1} + E_{3,2})$ must have term rank 2, a contradiction.

If T maps distinct collinear cells to distinct cells but a pair of noncollinear cells to the same cell, then, suppose that $T(E_{1,2}+E_{2,1}) = E_{1,2}$. Here we must have $T(E_{1,3})$, $T(E_{2,3})$, $T(E_{3,1})$ and $T(E_{3,2})$ are dominated by $R_1^{(0)} + C_2^{(0)}$, i.e., $T(K) \leq R_1^{(0)} + C_2^{(0)}$, a contradiction.

If $T(E_{1,2}+E_{3,1}) = E_{1,2}$ then $T(E_{1,3})$, $T(E_{3,2})$, and $T(E_{2,1})$ are all dominated by $R_1^{(0)} + C_2^{(0)}$. Unless $T(K)$ has a zero row or column we must have $T(E_{2,3}) = E_{2,1}$. But then $T(E_{2,1})$ must lie in the intersection of the two sets $\{E_{1,2}, E_{1,3}, E_{3,2}\}$ and $\{E_{2,1}, E_{2,3}, E_{3,1}\}$, which is empty, a contradiction.

Thus T does not map a pair of noncollinear cells to the same cell. It follows that T is one-to-one on the set of cells. Thus, by permuting and perhaps transposing, we may assume that $T(E_{1,2}) = E_{1,2}$ and $T(E_{1,3}) = E_{1,3}$. It is now easily seen that T is the identity transformation. That is, T is a (P, Q)-operator where P and Q are permutation matrices. If $Q \neq P^t$ then T does not map $\mathcal{M}_3(\mathbf{B})$ to itself, a contradiction. Thus T is a (P, P^t)-operator. ∎

Corollary 2.21 *Let* $T : \mathcal{M}_n(\mathbf{B})^{(0)} \to \mathcal{M}_n(\mathbf{B})^{(0)}$ *be an additive operator and* $T(0) = 0$*. If* T *preserves term rank-1 then either* T *is a* (P, P^t)*-operator, where* P *is a permutation matrix or* $T(A)$ *has a zero row or a zero column for all* $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$*.*

Proof. If $n = 2$, the corollary is trivially true since then the only transformations that preserve term rank-1 and such that $T(K)$ has no zero row or column are the identity and transpose transformations.

If $n = 3$, Lemma 2.20 establishes the corollary.

Now let us consider the case $n > 3$.

Note that in Theorem 2.18, if T is of the form (1) and either U or V is not a permutation matrix then $T(A)$ has a zero row or a zero column for any $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$.

Indeed, by conditions, the matrices U and V have no more than one entry 1 in each column (respectively, row). Thus each of them has at most n non-zero entries. Hence, if there is a row (column) with two non-zero entries, then there is a zero row (column).

Also if T is of the form (2) then $T(A)$ has a zero s-th row or a zero s-th column, respectively, for any $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$.

If T is of the form (3) with $i \neq j$ then $T(A)$ has a zero j-th row and a zero i-th column, respectively, for any $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$.

Note that by antinegativity of $\mathbf{B}$ we have that $T(A)$ has a zero row or a zero column for all $A \in \mathcal{M}_n(\mathbf{B})^{(0)}$ if and only if $T(K)$ has a zero row or column. Hence if $n > 3$ and if $T(K)$ does not have a zero row or column, then either: for some

i, $T(K) \leq R_i^{(0)} + C_i^{(0)}$; or T is a (U, V)-operator, where U and V are permutation matrices. If $V \neq U^t$, then $T(E_{i,j}) = E_{k,k}$ for some k and some $i \neq j$, a contradiction since the range of T does not contain $E_{k,k}$. If for some i, $T(K) \leq R_i^{(0)} + C_i^{(0)}$ and has no zero row or column, then there are indices (r, s) and (k, l) such that $T(E_{r,s}) \leq R_i^{(0)} \setminus C_i^{(0)}$ and $T(E_{k,l}) \leq C_i^{(0)} \setminus R_i^{(0)}$. But then for T to preserve term rank-1, $T(E_{r,l})$ must be $E_{i,i}$, an impossible case. ∎