Enumeration of coverings

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1. Introduction
2. Number of branched coverings
3. Number of orientable branched coverings
4. Distribution of branched coverings
5. Conclusion

1. Introduction

[EXA] The map $p : \mathbb{C} \to \mathbb{C}$ defined by $p(z) = z^n$ can be extended to a map $\tilde{p} : S^2 = \mathbb{C} \cup \{\infty\} \to S^2$ which is n to 1 except 0 and ∞.

[DEF] $\mathbb{S}$, $\widetilde{\mathbb{S}}$: surfaces, $B \subset \mathbb{S}$ with $|B| < \infty$.

- $f : \widetilde{\mathbb{S}} \to \mathbb{S}$ is a ($n-$fold) *branched covering*
  if $f|_{\widetilde{\mathbb{S}} - f^{-1}(B)} : \widetilde{\mathbb{S}} - f^{-1}(B) :\to \mathbb{S} - B$ is a ($n-$fold covering.)

- The smallest set B with this property = *branch set.*

[DEF] $f_1 : \widetilde{\mathbb{S}} \to \mathbb{S}$, $f_2 : \widetilde{\widetilde{\mathbb{S}}} \to \mathbb{S}$: two branched coverings f_1 and f_2 are equivalent if $\exists H$: homeo. such that

$$\begin{array}{ccc} \widetilde{\mathbb{S}} & \xrightarrow{H} & \widetilde{\widetilde{\mathbb{S}}} \\ & {\scriptstyle f_1} \searrow \quad \swarrow {\scriptstyle f_2} & \\ & \mathbb{S} & \end{array}$$

- A branched covering $f : \widetilde{\mathbb{S}} \to \mathbb{S}$ is *regular*
  if $\exists \mathcal{A}$: finite group and $\exists H$: homeo. such that

  $$\begin{array}{ccc} \widetilde{\mathbb{S}} & \xrightarrow{H} & \widetilde{\mathbb{S}}/\mathcal{A} \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle /\mathcal{A}} & \\ & \mathbb{S} & \end{array}$$

  We call it branched $\mathcal{A}-$covering.

- Every branched double covering is regular

[DEF]

$$\mathbb{S} = \begin{cases} S_k & \text{if } \mathbb{S} \text{ orientable surface of genus } k \\ N_h & \text{if } \mathbb{S} \text{ nonorientable surface of genus } h. \end{cases}$$

[FACT] (Alexander) Every orientable surface is a branched double covering surface of the sphere S_0.

[Note] The orientable surface S_k is a branched double covering of the sphere with $(2k+2)$ branch points & given such a branch set, the branched covering structure is unique up to equivalence.

[Question]

- Given surface $\mathbb{S}$, how many branched coverings are there?
- In how many different ways can a given surface be a branched covering of another given surface?

[Notations]

- $\forall\,\mathbb{S}$, $\mathbf{Isoc}\,(\mathbb{S}, B; n)$, $\mathbf{Isoc}^R(\mathbb{S}, B; n)$, $\mathbf{Isoc}\,(\mathbb{S}, B; \mathcal{A})$
- $\forall h > 0$, $\mathbf{Isoc}^O(N_h, B; n)$, $\mathbf{Isoc}^{OR}(N_h, B; n)$, $\mathbf{Isoc}^O(N_h, B; \mathcal{A})$
- $\forall\,\mathbb{S}, \forall\,\tilde{\mathbb{S}}$, $\mathbf{Iso}\,(\mathbb{S}, B; \tilde{\mathbb{S}}; n)$, $\mathbf{Iso}^R(\mathbb{S}, B; \tilde{\mathbb{S}}; n)$, $\mathbf{Iso}\,(\mathbb{S}, B; \tilde{\mathbb{S}}; \mathcal{A})$

[Note]

1. $\mathbf{Isoc}\,(\mathbb{S},\emptyset;n) = \#\{$conj. classes of subgps of index n in $\pi_1(\mathbb{S},*)\}$

2. $\mathbf{Isoc}^R(\mathbb{S},\emptyset;n) = \#\{$normal subgroups of index n in $\pi_1(\mathbb{S},*)\}$

3. $\mathbf{Isoc}\,(\mathbb{S},\emptyset;\mathcal{A}) = \#\{$normal subgroup $\mathcal{H}$ of $\pi_1(\mathbb{S},*)$ s.t. $\pi_1(\mathbb{S},*)/\mathcal{H} \cong \mathcal{A}\}$

4. $\mathbf{Iso}\,(\mathbb{S},B;\tilde{\mathbb{S}};n)$: (Generalized Hurwitz number)

2. Number of branched coverings

Given $\qquad \widetilde{\mathbb{S}}$

$\qquad\qquad \downarrow \quad f$: branched covering with branch set B.

$\qquad\qquad \mathbb{S}$

Let $* \in \mathbb{S} - B$ and $c \in B$. Let α_c be a counter clockwise oriented simple closed curve in $\mathbb{S} - B$ based at $*$, which is contractible in $\mathbb{S}$ and does not enclose any point in B except c. Then the n liftings of α_c induces a bijection $\widetilde{\alpha_c}$ on the preimage of the base point $*$.

$\Longrightarrow \mathcal{H}_f : \pi_1(\mathbb{S} - B, *) \to S_n$: gp. homo. s.t. $\widetilde{\alpha}_c \neq 1$,

$\forall$ branch pt. c (**Hurwitz system**)

Conversely, $h : \pi_1(\mathbb{S} - B, *) \to S_n$: Hurwitz system,

$\exists f_h : \widetilde{\mathbb{S}} \to \mathbb{S}$: branched covering s.t. $\mathcal{H}_{f_h} = h$.

(I) Base is Orientable

$$\pi_1(S_k - B, *) = \left\{ a_i, b_i, c_j \;\middle|\; \prod_{i=1}^{k} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{|B|} c_j = 1 \right\}$$

$\mathcal{H}_{S_k}(B, n)$: $(\sigma_1, \tau_1, \ldots, \sigma_k, \tau_k, \mu_1 \ldots, \mu_{|B|}) \in (S_n)^{2k+|B|}$ s.t.

(1) $< \sigma_1, \tau_1, \ldots, \sigma_k, \tau_k, \mu_1 \ldots, \mu_{|B|} >$ acts transitively on $\{1, 2, \ldots, n\}$

(2) $\sigma_1 \tau_1 \sigma_1^{-1} \tau_1^{-1} \cdots \sigma_k \tau_k \sigma_k^{-1} \tau_k^{-1} \mu_1 \cdots \mu_{|B|} = 1$

(3) $\mu_\ell \neq 1 \ \forall \ell = 1, \ldots, |B|$.

(II) Base is nonorientable

$$\pi_1(N_h - B, *) = \left\{ a_i, c_j \;\middle|\; \prod_{i=1}^{h} a_i^2 \prod_{j=1}^{|B|} c_j = 1 \right\}$$

$\mathcal{H}_{N_h}(B, n) : (\sigma_1, \dots, \sigma_h, \mu_1 \dots, \mu_{|B|}) \in (S_n)^{h+|B|}$ s.t.

(1) $< \sigma_1, \dots, \sigma_h, \mu_1 \dots, \mu_{|B|} >$ acts transitively on $\{1, 2, \dots, n\}$

(2) $\sigma_1^2 \cdots \sigma_h^2 \mu_1 \cdots \mu_{|B|} = 1$

(3) $\mu_\ell \neq 1$, $\forall \ell = 1, \dots, |B|$.

Similarly, we can define $\mathcal{H}_{\mathbb{S}}(B, \mathcal{A})$, $\forall\, \mathbb{S}$.

[THM]

1. $\mathbf{Isoc}\,(\mathbb{S}, B; n) = |\mathcal{H}_{\mathbb{S}}(B, n)/S_n|$
   coordinatewise conjugacy action.

2. $\mathbf{Isoc}\,(\mathbb{S}, B; \mathcal{A}) = |\mathcal{H}_{\mathbb{S}}(B, \mathcal{A})/\mathsf{Aut}(\mathcal{A})|$
   coordinatewise action.

3. $\mathbf{Isoc}^R(\mathbb{S}, B; n) = \sum_{\mathcal{A}} \mathbf{Isoc}\,(\mathbb{S}, B; \mathcal{A})$

[Note] $\mathbf{Isoc}\,(S_k, \emptyset; n)(\forall k \geq 0)$ was computed by Mednykh(1979)

$\mathbf{Isoc}\,(N_h, \emptyset; n)(\forall h > 0)$ was computed by
Mednykh & Pozdnyakova(1986)

Method; Burnside's Lemma & character theory on S_n

In general, Mednykh(2006) found a counting formula for the number of conjugacy classes of subgroups of given index in a finitely generated group.

THM] (Mednykh, Kwak, Lee : 2003)

$$\mathbf{Isoc}\,(\mathbb{S}, B; n) = (-1)^b \mathbf{Isoc}\,(\mathbb{S}, \emptyset; n) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}\,(\mathfrak{B}_{\alpha_{\mathbb{S}}+b-t-1}; n),$$

$$\mathbf{Isoc}\,(\mathbb{S}, B; \mathcal{A}) = (-1)^b \mathbf{Isoc}\,(\mathbb{S}, \emptyset; \mathcal{A}) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}\,(\mathfrak{B}_{\alpha_{\mathbb{S}}+b-t-1}; \mathcal{A}),$$

where $b = |B|$, $\alpha_{\mathbb{S}} = 2k,\ \forall \mathbb{S} = S_k$ and $\alpha_{\mathbb{S}} = h,\ \forall \mathbb{S} = N_h$.

Pf. Principle of inclusion & exclusion

[Note] $\mathbf{Isoc}\,(G; n)$ was computed by many authors:

Hofmeister, Kwak & Lee, Liskovets, Mednykh, ...

[Exa] $\forall k$, $\mathbf{Isoc}\,(S_k, B; 3) = 6^{2k-2}(5^b+(-1)^b)+3^{2k-2}(2^b+(-1)^b 3)$
$-2^{2k-2}(1+(-1)^b)$

[THM] (Jones(1999) + GKL(2004))

$$\mathbf{Isoc}\,(S_k, B; \mathcal{A}) = \sum_{K \leq \mathcal{A}} \frac{\mu(K)|K|^{2k-1}}{|\mathrm{Aut}(\mathcal{A})|} \left((|K|-1)^b + (-1)^b \sum_{\xi \neq Pr} \xi(1)^{2-2k} \right)$$

$$\mathbf{Isoc}\,(N_h, B; \mathcal{A}) = \sum_{K \leq \mathcal{A}} \frac{\mu(K)|K|^{h-1}}{|\mathrm{Aut}(\mathcal{A})|} \left((|K|-1)^b + (-1)^b \sum_{\xi \neq Pr} c_\xi^h \, \xi(1)^{2-h} \right),$$

where μ is the set theoretical Möbius mu-function and $c_\xi = 1, -1, 0$ if ρ is real, ξ is real but ρ is not real, ξ is not real.

[Note] From this, we can have an explicit formula for $\mathbf{Isoc}^R(\mathbb{S}, B; n)$ when $n = p, 2p, p^2$.

3. Number of orientable branched coverings

[Note] $\forall k \geq 0$, $\mathbf{Isoc}^O(S_k, B; n) = \mathbf{Isoc}\,(S_k, B; n)$.

$\forall h > 0$, $\mathbf{Isoc}^O(N_h, B; n) \neq 0 \Longrightarrow n$: even

- $H^0_{N_h}(B; 2n); (\sigma_1, \ldots, \sigma_h, \mu_1, \ldots, \mu_{|B|}) \in H_{N_h}(B; 2n)$ s.t. σ_s reverses parity and μ_t preserves parity.

- $H^0_{N_h}(B; \mathcal{A}); (\sigma_1, \ldots, \sigma_h, \mu_1, \ldots, \mu_b) \in H_{H_h}(B; \mathcal{A})$ s.t. $\sigma_1, \ldots, \sigma_h \in \mathcal{A} - S$ and $\mu_1, \ldots, \mu_b \in S$ for some subgroup S of index 2 in $\mathcal{A}$.

[THM] Let $h > 0$. Then we have

$$\mathbf{Isoc}^{O}(N_h, B; n) = |H^{0}_{N_h}(B; 2n)/(P_{2n} \cup R_{2n})|,$$

coordinatewise conjugacy action.

$$\mathbf{Isoc}^{OR}(N_h, B; n) = \sum_{\mathcal{A}} \mathbf{Isoc}^{O}(N_h, B; \mathcal{A}) \text{ and}$$

$$\mathbf{Isoc}^{O}(N_h, B; \mathcal{A}) = |H^{0}_{N_h}(B; \mathcal{A})/\mathsf{Aut}(\mathcal{A})|,$$

coordinatewise action.

THM] (Mednykh, Kwak, Lee : 2003)

$$\mathbf{Isoc}^O(N_h, B; n) = (-1)^b \mathbf{Isoc}^O(N_h, \emptyset; 2n) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}^B(\mathfrak{B}_{h+b-t-1}; 2n),$$

$$\mathbf{Isoc}^O(N_h, B; \mathcal{A}) = (-1)^b \mathbf{Isoc}^O(N_h, \emptyset; \mathcal{A}) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}^B(\mathfrak{B}_{h+b-t-1}; \mathcal{A}),$$

[Note] $\mathbf{Isoc}^O(N_h, B; 2n)$ was computed by Goulden, Kwak & Lee (2005)

For example: $\mathbf{Isoc}^O(N_h, \emptyset; 2p) = \lceil \frac{p+3}{2} \rceil$. ($p$ prime)

	N_1	N_2	N_3	N_4
$n=1$	1	1	1	1
$n=2$	0	3	9	39
$n=3$	0	3	57	1483
$n=4$	0	6	847	354009
$n=5$	0	4	15303	208211284

The number $\mathbf{Isoc}^O(N_k, \emptyset; 2n)$

[Note] $\mathbf{Isoc}^O(N_h, B; \mathcal{A})$ ($\mathcal{A}$: abelian) was computed by

Kwak, Lee & Shin (2004).

It does not involve character theory

$\mathbf{Isoc}^{OR}(N_h, B; n)$ was computed by

Goulden, Kwak & Lee (2004)

(**Method**: character theory & Burnside's Lemma)

[THM] 1. $\forall k \geq 0$, $\mathbf{Isoc}^O(S_k, B; \mathcal{A}) = \mathbf{Isoc}\,(S_k, B; \mathcal{A})$

2. $\forall h > 0$, $\mathbf{Isoc}^O(N_h, B; \mathcal{A})$ is equal to

$$\sum_{\mathcal{S}} \sum_{K \leq (\mathcal{A},\mathcal{S})} \frac{\mu(K)\,|K|^{h-1}}{2^{h-1}|\mathrm{Aut}(\mathcal{A},\mathcal{S})|} \left(\left(\frac{|K|}{2} - 1 \right)^b + (-1)^b \sum_{\xi} d_\xi^h\, \xi(1)^{2-h} \right),$$

where $\mathcal{S} \in \{\mathcal{S} \mid [\mathcal{A} : \mathcal{S}] = 2\}/\!\sim$, $\xi \in \mathrm{irr}(K \cap \mathcal{S}) - \{Pr\}$.

[EXA] $\mathbf{Isoc}^{OR}(\mathbb{S}, B; 2p)$

$$= \begin{cases} \mathbf{Isoc}^{R}(S_k, B; 2p) & \text{if } \mathbb{S} = S_k, \\ \dfrac{2}{p-1}\left(p^{h-1} - 1\right) & \text{if } \mathbb{S} = N_h \text{ and } b = 0, \\ p^{h-2}\left((p-1)^{b-1}(p+1) + (-1)^{b}\right) & \text{if } \mathbb{S} = N_h \text{ and } b \neq 0. \end{cases}$$

4. Distribution of branched coverings

[Note]

- $\mathbf{Iso}\,(S_i, B; S_j; n)$ was computed by Mednykh (1984)

  Method: character theory on S_n and Buside's Lemma,
  Riemann-Hurwite equation

  (**It is not explicit !!**)

- $\mathbf{Iso}^R(\mathbb{S}, B; \widetilde{\mathbb{S}}; n) = \sum_{\mathcal{A}} \mathbf{Iso}\,(\mathbb{S}, B; \widetilde{\mathbb{S}}; \mathcal{A})$

- $\mathbf{Iso}\,(S_i, B; S_j; \mathcal{A})$ was computed by Jones (1999).

- $\mathbf{Iso}\,(N_h, B; S_k; \mathcal{A})$ and $\mathbf{Iso}\,(N_h, B; N_k; \mathcal{A})$
  was computed by Goulden, Kwak & Lee (2004)

  (**They are not explicit !!**)

n	$\lvert B\rvert = 2$	$\lvert B\rvert = 3$	$\lvert B\rvert = 4$
4	$120\,x^7 + 256\,x^8$	$768\,x^9$	$120\,x^9 + 1536\,x^{10} + 1024\,x^{11}$
6	$640\,x^{10} + 1215\,x^{11} + 1296\,x^{12}$	$8991\,x^{13} + 3888\,x^{14}$	$640\,x^{13} + 7776\,x^{14} + 26973\,x^{15} + 28512\,x^{16} + 3888\,x^{17}$
7	$2401\,x^{14}$	$12005\,x^{17}$	$74431\,x^{20}$
8	$960\,x^{13} + 1920\,x^{15} + 4096\,x^{16}$	$5760\,x^{17} + 12288\,x^{18} + 24576\,x^{19}$	$960\,x^{17} + 11520\,x^{19} + 24576\,x^{20} + 105984\,x^{21} + 98304\,x^{22} + 65536\,x^{23}$
9	$2160\,x^{16} + 6561\,x^{18}$	$2160\,x^{19} + 39366\,x^{21} + 19683\,x^{22}$	$6480\,x^{22} + 157464\,x^{24} + 157464\,x^{25} + 177147\,x^{26}$
10	$2496\,x^{16} + 9375\,x^{19} + 10000\,x^{20}$	$60000\,x^{22} + 28125\,x^{23} + 90000\,x^{24}$	$2496\,x^{21} + 60000\,x^{24} + 60000\,x^{25} + 360000\,x^{26} + 241875\,x^{27} + 780000\,x^{28} + 130000\,x^{29}$
12	$4800\,x^{19} + 9720\,x^{21} + 10240\,x^{22} + 9720\,x^{23} + 20736\,x^{24}$	$98760\,x^{25} + 215784\,x^{27} + 186624\,x^{28} + 62208\,x^{29}$	$4800\,x^{25} + 58320\,x^{27} + 61440\,x^{28} + 204120\,x^{29} + 746496\,x^{30} + 1498960\,x^{31} + 1244160\,x^{32} + 1522152\,x^{33} + 705024\,x^{34} + 248832\,x^{35}$

$\mathbf{Iso}\,(S_2, B; S_j; \mathbb{Z}_n)$ for small n and small $|B|$

n	$\lvert B\rvert = 2$	$\lvert B\rvert = 3$	$\lvert B\rvert = 4$
4	$4\,x^{-4}$	$x^2 + 12\,x^{-6}$	$24\,x^{-8} + 16\,x^{-10}$
6	$2\,x^2 + 4\,x^{-4} + 4\,x^{-6}$	$4\,x^4 + 6\,x^{-6} + 24\,x^{-8} + 24\,x^{-10}$	$8\,x^6 + 32\,x^{-10} + 120\,x^{-12} + 128\,x^{-14} + 16\,x^{-16}$
7	$6\,x^{-7}$	$36\,x^{-13}$	$216\,x^{-19}$
8	$2\,x^2 + 8\,x^{-8}$	$3\,x^4 + 4\,x^6 + 24\,x^{-12} + 48\,x^{-14}$	$4\,x^6 + 16\,x^8 + 48\,x^{-16} + 192\,x^{-18} + 192\,x^{-20} + 128\,x^{-22}$
9	$4\,x^{-7} + 6\,x^{-9}$	$12\,x^{-13} + 36\,x^{-15} + 36\,x^{-17}$	$32\,x^{-19} + 144\,x^{-21} + 288\,x^{-23} + 216\,x^{-25}$
10	$4\,x^4 + 4\,x^{-6} + 8\,x^{-10}$	$16\,x^8 + 6\,x^{-10} + 48\,x^{-14} + 96\,x^{-18}$	$64\,x^{12} + 8\,x^{-16} + 48\,x^{-18} + 48\,x^{-20} + 384\,x^{-22} + 128\,x^{-24} + 768\,x^{-26} + 128\,x^{-28}$
12	$2\,x^2 + 4\,x^4 + 8\,x^{-10} + 8\,x^{-12}$	$9\,x^6 + 18\,x^8 + 4\,x^{10} + 36\,x^{-16} + 84\,x^{-18} + 96\,x^{-20} + 48\,x^{-22}$	$4\,x^8 + 40\,x^{10} + 80\,x^{12} + 32\,x^{14} + 96\,x^{-22} + 336\,x^{-24} + 672\,x^{-26} + 880\,x^{-28} + 768\,x^{-30} + 448\,x^{-32} + 128\,x^{-34}$

$\mathbf{Iso}\,(N_1, B; \mathbb{S}; \mathbb{Z}_n)$ for small n and small $\lvert B\rvert$

$\mathcal{A}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{D}_4$	the others	total
$\lvert B\rvert = 0$	0	0	0	0	0	0	0	0
$\lvert B\rvert = 1$	0	0	3	2	2	2	0	9
$\lvert B\rvert = 2$	2	2	0	0	0	0	0	4
$\lvert B\rvert \geq 3$	0	0	0	0	0	0	0	0
total	2	2	3	2	2	2	0	13

The number $\mathbf{Iso}\,(N_2, B; S_3; \mathcal{A})$

$$\mathbf{Iso}^R(N_2, B; S_3; n) = \begin{cases} 3 & \text{if } n = 6 \text{ and } |B| = 1, \\ 6 & \text{if } n = 8 \text{ and } |B| = 1, \\ 4 & \text{if } n = 4 \text{ and } |B| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

5. Conclusion

[Problem 1] Count various types of surface branched coverings. (e.g., Simple ones, etc.)

[Problem 2] $\forall h, \forall k, \forall n \ \mathbf{Iso}\,(N_h, B; S_k; n) =?$

[Problem 3] $f : \widetilde{\mathbb{S}} \to \mathbb{S}, g : \widetilde{\widetilde{\mathbb{S}}} \to \mathbb{S}$: branched coverings

$$f \simeq_w g \iff \begin{array}{ccc} \widetilde{\mathbb{S}} & \xrightarrow{\exists\text{Homeo.}} & \widetilde{\widetilde{\mathbb{S}}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{S} & \xrightarrow[\exists\text{Homeo.}]{} & \mathbb{S} \end{array}$$

Count the weak equivalence classes of (regular) branched coverings over a given surface.

[FACT] $\mathcal{A}$: finite group, acts pseudofreely on $\mathbb{S}$ if

$$|\{x \in \mathbb{S} : g \cdot x = x\}| < \infty$$

for any $g \in \mathcal{A}$.

$\Longrightarrow$ # weak equivalence classes of pseudofree actions on $\mathbb{S}$ is

$$\sum_{k=0}^{\infty} \sum_{B \subset S_k} \mathbf{Isow}(S_k, B; \mathbb{S}; \mathcal{A}) + \sum_{h=1}^{\infty} \sum_{B \subset N_h} \mathbf{Isow}(N_h, B; \mathbb{S}; \mathcal{A}).$$

$\sum_{B \subset \mathbb{S}} \mathbf{Isow}(\mathbb{S}, B; \widetilde{\mathbb{S}}; \mathcal{A})=$ # of weak equivalence classes of pseudofree actions on $\widetilde{\mathbb{S}}$ such that $\widetilde{\mathbb{S}}/\mathcal{A} \simeq \mathbb{S}$

[Note]

- $\mathbf{Isow}(S^2, B; \mathbb{S}; \mathbb{Z}_p)$ was computed by Kwak & Lee (1996).
- $\mathbf{Isow}(S^2, B; \mathbb{S}; \mathbb{Z}_p \times \mathbb{Z}_q)$ was computed by Lee & Kim (2002).

Thank you!