Learning today, leading tomorrow

Numerical Algorithm for solving (non) linear problems with Taylor series

Bongsoo Jang

School of Mechanical & Advanced Materials & Graduate School of Natural Sciences

Ulsan National University of Science and Technology(UNIST)

http://amath.unist.ac.kr

bsjang@unist.ac.kr

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Outline

- Introduction to the Differential Transform Method(DTM)
- Performance of DTM : Schrodinger Equations
- Difficulties and Enhanced DTM
  - Multistage DTM and Examples
  - Projected DTM and Examples
  - Boundary DTM and Examples
- Challenging problems : Fractional DTM and Examples
- Summary and conclusion

Taylor Series

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k u(x_0)}{dx^k} (x - x_0)^k$$

•Brook Taylor (1685-1731)

Introduction to the DTM

Differential Transform Method(DTM)

- Numerical Method using Taylor Series

Suppose a function u(x) is analytic in (0,1) and $x_0 \in (0,1)$

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k u(x_0)}{dx^k} (x - x_0)^k$$

U(k): the differential transform of u(x) at $x = x_0$ **defined by**

$$U(k) = \frac{1}{k!} \frac{d^k u(x_0)}{dx^k}$$

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k u(x_0)}{dx^k} (x - x_0)^k = \sum_{k=0}^{\infty} U(k)(x - x_0)^k$$

Example

$$\frac{d^2u}{dx^2}+\frac{du}{dx}+u=f(x),u(0)=1,\frac{du(0)}{dx}=-1$$

$$\frac{d^2}{dx^2}(\sum_{k=0}^{\infty}U(k)x^k)+\frac{d}{dx}(\sum_{k=0}^{\infty}U(k)x^k)+(\sum_{k=0}^{\infty}U(k)x^k)=(\sum_{k=0}^{\infty}F(k)x^k)$$

$$\sum_{k=0}^{\infty}((k+1)(k+2)U(k+2))x^k+\sum_{k=0}^{\infty}(k+1)U(k+1)x^k+\sum_{k=0}^{\infty}U(k)x^k=\sum_{k=0}^{\infty}F(k)x^k$$

- **Collecting all coefficients of** x^k

$$(k+1)(k+2)U(k+2)+(k+1)U(k+1)+U(k)=F(k)$$

- **From initial conditions**

$$U(0)=1,U(1)=-1$$

Properties of DTM

- **Taylor Series of u(x) at x=0**

- **Derivative**

$$\frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty} U(k)x^k\right) = \frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty} \frac{1}{k!}\frac{du^k(0)}{dx^k}x^k\right) = \sum_{k=2}^{\infty} \frac{1}{k!}\frac{du^k(0)}{dx^k}k(k-1)x^{k-2}$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{(k+2)!}\frac{du^{k+2}(0)}{dx^{k+2}}x^k = \sum_{k=0}^{\infty}(k+1)(k+2)U(k+2)x^k$$

Suppose $w(x) = \dfrac{d^r}{dx^r}u(x)$

$$W(k) = (k+1)(k+2)\cdots(k+r)U(k+r) = \frac{(k+r)!}{k!}U(k+r)$$

Properties of DTM

- **Product**

$$\left(\sum_{i=0}^{\infty} U(i)x^i\right)\left(\sum_{j=0}^{\infty} U(j)x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{r=0}^{k} U(r)U(k-r)\right)x^k$$

Suppose $w(x) = u(x)v(x)$

$$W(k) = \sum_{r=0}^{k} U(r)U(k-r)$$

Suppose $w(x) = \dfrac{du(x)}{dx}\dfrac{dv(x)}{dx}$

$$W(k) = \sum_{r=0}^{k} (r+1)(k-r+1)U(r+1)U(k-r+1)$$

Introduction to the Multidimensional DTM

Suppose a function u(x,y,t) is analytic in (0,1) and $(x_0, y_0, z_0) \in (0,1)$

U(k,h,m): the differential transform of u(x) at (x_0, y_0, z_0) **defined by**

$$U(k,h,m) = \frac{1}{k!h!m!} \frac{\partial^{k+h+m} u(x_0, y_0, z_0)}{\partial x^k \partial y^h \partial z^m}$$

U(x,y,z): the differential inverse transform of U(k,h,m)

$$u(x,y,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} U(k,h,m)(x-x_0)^k (y-y_0)^h (z-z_0)^m$$

Properties of Multidimensional DTM

Table 1
Fundamental operations for the three dimensional DTM

Original function $w(x,y,t)$	Transformed function $W(k,h,m)$
$u(x,y,t) \pm v(x,y,t)$	$U(k,h,m) \pm V(k,h,m)$
$cu(x,y,t)$	$cU(k,h,m)$
$\frac{\partial^{r+s+p} u(x,y,t)}{\partial x^r \partial y^s \partial t^p}$	$\frac{(k+r)!}{k!}\frac{(h+s)!}{s!}\frac{(m+p)!}{p!}U(k+r,h+s,m+p)$
$\frac{\partial u(x,y,t)}{\partial x}\frac{\partial v(x,y,t)}{\partial y}$	$\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)(h-s+1)U(k-r+1,s,p)U(r,h-s+1,m-p)$
$u(x,y,t)v(x,y,t)$	$\sum_{r=0}^{j}\sum_{s=0}^{h}\sum_{p=0}^{m}U(r,h-s,m-p)V(k-r,s,p)$
$u(x,y,t)v(x,y,t)g(x,y,t)$	$\sum_{r=0}^{k}\sum_{\tau=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{l=0}^{m}\sum_{q=0}^{m-l}U(r,h-s-p,l)V(\tau,s,m-l-q)G(k-r-\tau,p,q)$

Examples

Example 1 (Nonlinear Schrodinger equation)

$$iu_t + u_{xx} + 2|u|^2 u = 0, \qquad u(x,0) = exp(ix),$$

$$i(h+1)U(k,h+1) + (k+1)(k+2)U(k+2,h)$$
$$+2\sum_{r=0}^{k}\sum_{\tau=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s} U(r,h-s-p)\overline{U(\tau,s)}U(k-r-\tau,p) = 0$$

Initial Condition:

$$U(k,0) = \frac{i^k}{k!}$$

Examples

$$u(x,t) = \sum_{k=0}^{\infty} \frac{i^k}{k!} x^k + i\left(\sum_{k=0}^{\infty} \frac{i^k}{k!} x^k\right) t - \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{i^k}{k!} x^k\right) t^2 - \frac{i}{6}\left(\sum_{k=0}^{\infty} \frac{i^k}{k!} x^k\right) t^3 + \cdots$$

$$= \exp(ix) + i\exp(ix) - \frac{1}{2}\exp(ix)t^2 - \frac{i}{6}\exp(ix)t^3 + \cdots$$

$$= \exp(i(x+t)).$$

Examples

Example 2 (Nonlinear Schrodinger equation with trapping potential)

$$iu_t + \frac{1}{2}\Delta u - u\psi - |u|^2 u = 0,$$
$$\psi(x,y) = 1 - \sin^2 x \sin^2 y, u(x,y,0) = \sin x \sin y,$$

$$i(m+1)U(k,h,m+1) + \frac{1}{2}[(k+1)(k+2)U(k+2,h,m) + (h+1)(h+2)U(k,h+2,m)$$
$$-\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m} U(r,h-s,m-p)\Psi(k-r,s,p)$$
$$-\sum_{r=0}^{k}\sum_{\tau=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}\sum_{l=0}^{m}\sum_{q=0}^{m-l} U(r,h-s-p,l)\overline{U(\tau,s,m-l-q)}U(k-r-\tau,p,q) = 0.$$

Examples

Initial Condition:

$$U(2k+1,2h+1,0)=\frac{(-1)^{k+h}}{(2k+1)!(2h+1)!},U(2k,2h+1)=U(2k+1,2h,0)=0$$

From $\psi(x,y)=1-\sin^2 x\sin^2 y,$ **we have**

$$\Psi(0,0,0)=1,\Psi(2,0,0)=-1,\Psi(4,0,0)=1/3,$$
$$\Psi(2,2,0)=0,\Psi(4,2,0)=-1/3,\Psi(6,2,0)=2/45,$$
$$\Psi(k,2h+1,0)=0,\Psi(2k+1,h,0)=0,$$

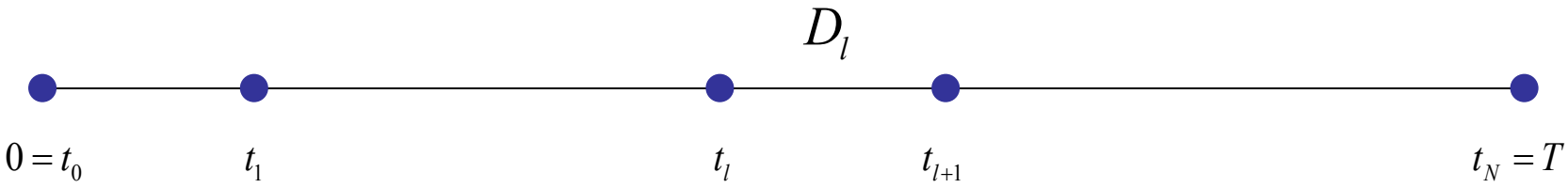
Examples

$$
\begin{aligned}
&u(x,t)\\
&= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^k\right) y - \frac{1}{6}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^k\right) y^3 + \cdots\\
&+ \left\{-2i\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^k\right) y + \frac{i}{3}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^k\right) y^3 + \cdots\right\} t\\
&+ \left\{-2\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^k\right) y + \frac{1}{3}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^k\right) y^3 + \cdots\right\} t^2 + \cdots\\
&= \sin x \left[\left(\sum_{h=0}^{\infty} \frac{(-1)^h}{(2h+1)!}y^k\right) - 2i\left(\sum_{h=0}^{\infty} \frac{(-1)^h}{(2h+1)!}y^k\right) t - 2\left(\sum_{h=0}^{\infty} \frac{(-1)^h}{(2h+1)!}y^k\right) t^2 + \cdots\right],
\end{aligned}
$$

which gives directly the exact solution

$$u(x,y,t) = \sin x \sin y \exp(-2it).$$

Difficulties and Multistage DTM



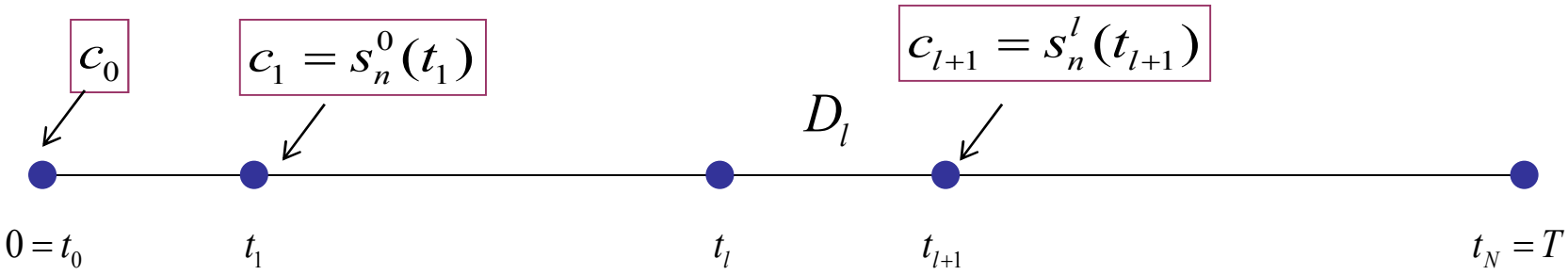
$X^l(k)$ **: the differential transform of x(t) at** $t = t_l$ **defined by**

$$X^l(k) = \frac{1}{k!}\frac{d^k x^l(t_0)}{dt^k}$$

$x^l(t)$ **: the differential inverse transform of** $X^l(k)$

$$x^l(t) = \sum_{k=0}^{\infty} X^l(k)(t - t_l)^h$$

Multistage DTM: Update condition (MDTM)



Update conditions:

(1) $$x^l(t) \approx \sum_{k=0}^{n} X^l(k)(t-t_l)^h \equiv s_n^l(t)$$

(2) $$X^{l+1}(0) = s_n^l(t_{l+1})$$

Examples of Multistage DTM

Example (Lotka-Volterra equation with m species)

$$\frac{d}{dt}x_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{m} a_{ij}x_j(t)\right), x_i(0) = c_i, i = 1,2,\cdots m.$$

$$(k+1)X_i^l(k+1) = b_i X_i^l(k) + \sum_{j=1}^{m} a_{ij}\left(\sum_{p=0}^{k} X_i^l(p)X_i^l(k-p)\right), i = 1,2,\cdots m.$$

Initial Condition:

$$X_i^0(0) = c_i, X_i^l(0) = s_{i,n}^{l-1}(t_l).$$

Examples of Multistage DTM

Table 3
Numerical results of two species with $a_{11} = -0.0014, a_{12} = -0.0012, a_{21} = -0.0009, a_{22} = -0.001, b_1 = 0.1, b_2 = 0.08$.

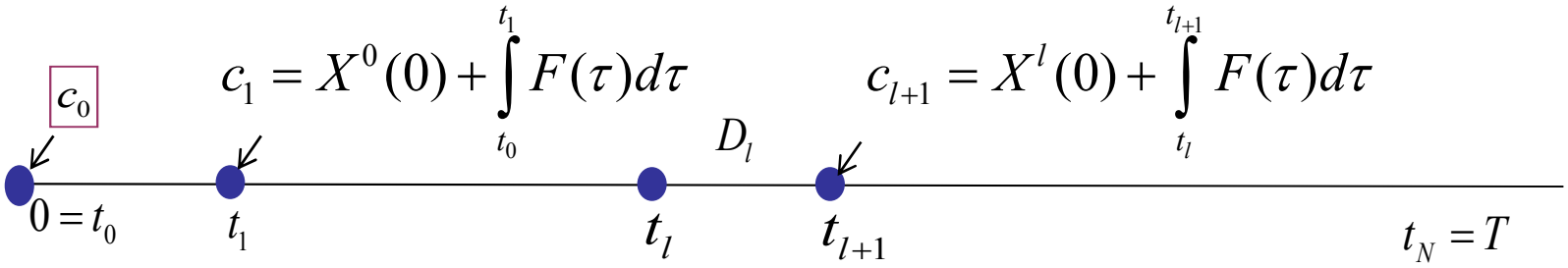
	$x_1(t)$			$x_2(t)$		
	RK4	MVIM	MDTM	RK4	MVIM	MDTM
$t \setminus h$	0.001	0.01	0.1	0.001	0.01	0.1
2	4.7043	4.7045	4.7043	11.398	11.398	11.397
4	5.5016	5.5020	5.5016	12.935	12.935	12.934
6	6.3942	6.3949	6.3941	14.610	14.611	14.610
8	7.3811	7.3823	7.3810	16.417	16.418	16.416
10	8.4576	8.4598	8.4576	18.344	18.347	18.344
12	9.6152	9.6180	9.6151	20.376	20.380	20.376
14	10.841	10.845	10.840	22.490	22.496	22.491
16	12.117	12.122	12.116	24.663	24.670	24.663
18	13.423	13.432	13.423	26.864	26.874	26.864
20	14.738	14.750	14.738	29.065	29.078	29.064

Examples of Multistage DTM

Table 4
Numerical results of three species with $a_{11} = -1, a_{12} = a_{13} = -0.1, a_{21} = -0.1, a_{22} = -1, a_{23} = -0.1, a_{31} = -0.1, a_{32} = -0.1, a_{33} = -1, b_1 = b_2 = b_3 = 1$.

	$x_1(t)$			$x_2(t)$			$x_3(t)$		
	RK4	MVIM	MDTM	RK4	MVIM	MDTM	RK4	MVIM	MDTM
$t \setminus h$	0.001	0.01	0.1	0.001	0.01	0.1	0.001	0.01	0.1
0.5	0.2797	0.2837	0.2797	0.3993	0.4042	0.3993	0.6067	0.6126	0.6067
1.0	0.3711	0.3764	0.3711	0.5005	0.5073	0.5006	0.6944	0.7039	0.6944
1.5	0.4656	0.4733	0.4557	0.5926	0.6034	0.5927	0.7580	0.7776	0.7579
2.0	0.5541	0.5677	0.5542	0.6678	0.6810	0.6679	0.7990	0.8479	0.7990
2.5	0.6297	0.6598	0.6298	0.7242	0.7785	0.7242	0.8229	0.9512	0.8228
3.0	0.6898	0.7656	0.6896	0.7635	0.9052	0.7635	0.8352	1.1630	0.8350

Multistage DTM: Update condition (MDTM I)



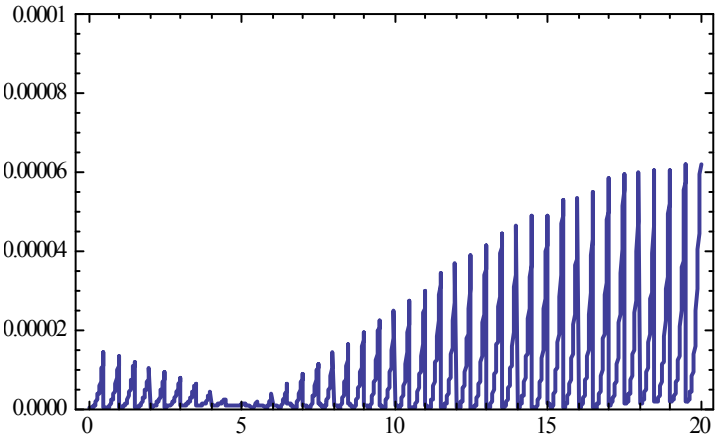
Update conditions:

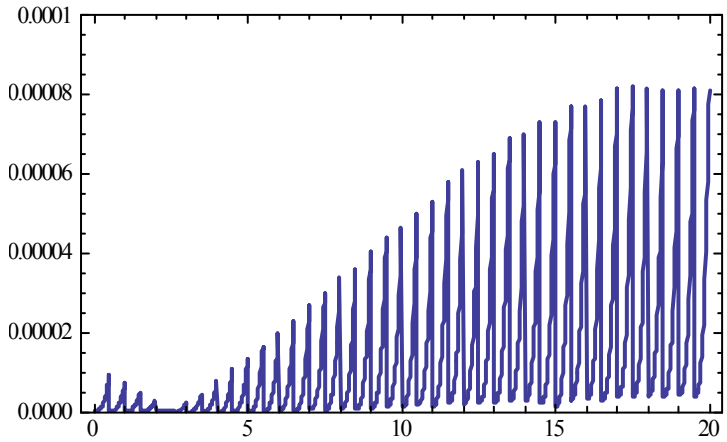
(1) $$\frac{dx^l}{dt} = f(t, x^l) \Longrightarrow x^l(t) = x^l(t_l) + \int_{t_l}^{t} f(\tau, x^l(\tau))d\tau$$

(2) $$x^l(t) \approx x^l(t_l) + \int_{t_l}^{t} f(\tau, s_n^l(\tau))d\tau$$

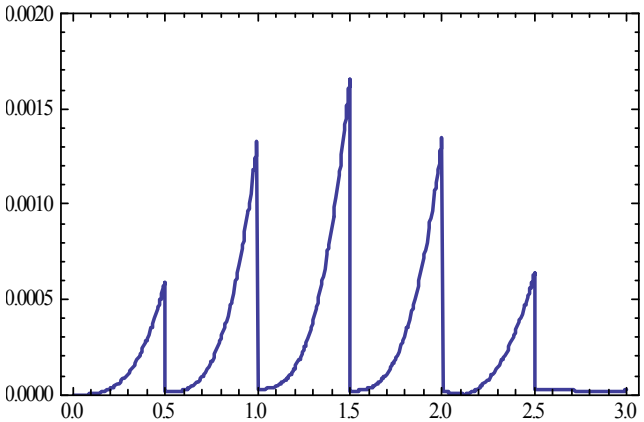
(3) $$X^{l+1}(0) = X^l(0) + \int_{t_l}^{t_{l+1}} F(\tau)d\tau$$

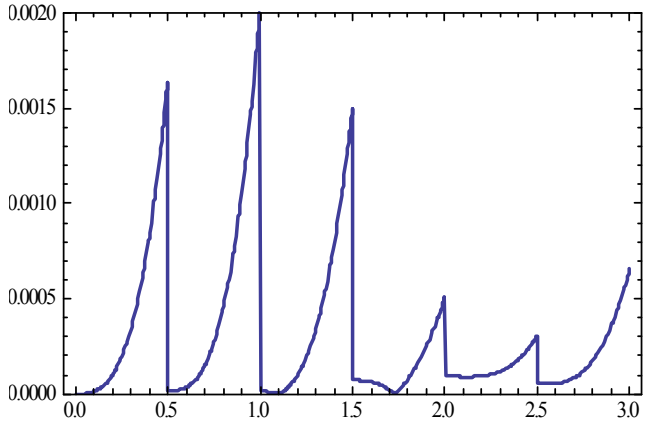
Error: MDTM I(h=0.5) and RK4(h=0.001)

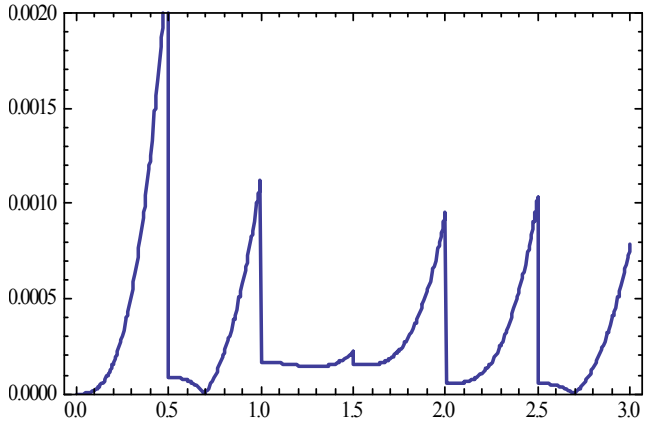




Two species: $x_1(t), x_2(t)$



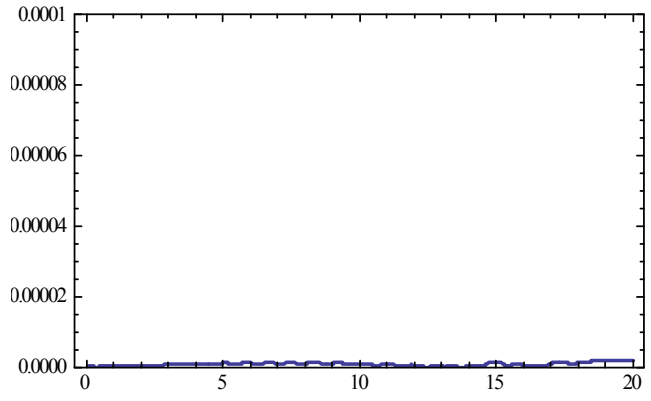


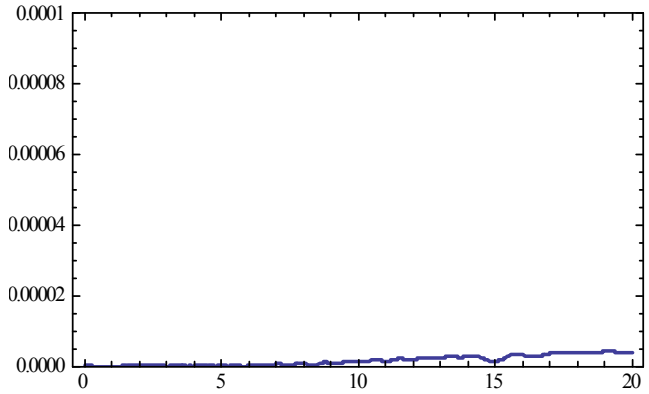


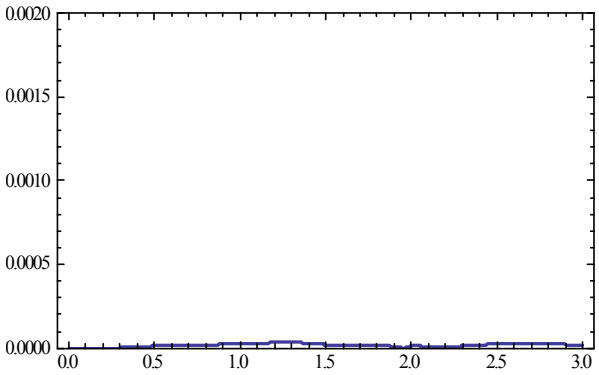
Three species: $x_1(t), x_2(t), x_3(t)$

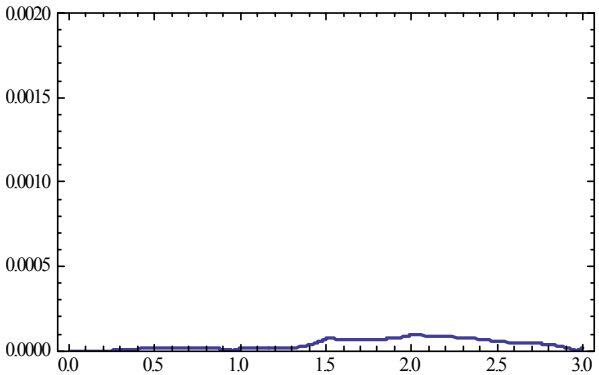
Multistage DTM: Update condition (MDTM II)

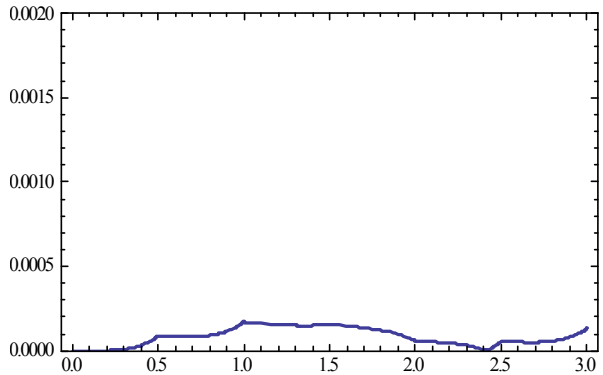
$$x^l(t) \approx x^l(t_l) + \int_{t_l}^{t} f(\tau, s_n^l(\tau))d\tau$$











Difficulties: Complex nonlinearity

$$\sin u = \sin(\sum_{k=0}^{\infty} U(k)(x-x_0)^k) = \sum_{k=0}^{\infty} \widehat{U}(k)(x-x_0)^k$$

$$f(x) = \sin(u(x)), g(x) = \cos(u(x))$$

$$\Rightarrow \frac{d}{dx} f(x) = \cos(u(x)) \frac{d}{dx} u(x) = g(x) \frac{d}{dx} u(x)$$

$$\Rightarrow \frac{d}{dx} g(x) = -\sin(u(x)) \frac{d}{dx} u(x) = -f(x) \frac{d}{dx} u(x),$$

Complex nonlinearity

$$(k+1)F(k+1) = \sum_{r=0}^{k}(k-r+1)G(r)U(k-r+1)$$

$$(k+1)G(k+1) = -\sum_{r=0}^{h}(k-r+1)F(r)U(k-r+1)$$

$$F(x,0) = \sin(U(x,0)), G(x,0) = \cos(U(x,0))$$

Difficulties and Projected DTM

$U(x,h)$: the projected DT of $u(x,t)$ with respect to $t = t_l$

$$U(x,h) = \frac{1}{h!}\left[\frac{\partial^h}{\partial t^h} u(x,t_l)\right]$$

$u(x,t)$: the projected differential inverse transform of $U(x,h)$

$$u(x,t) = \sum_{h=0}^{\infty} U(x,h)(t-t_l)^h$$

Projected DTM: Complex nonlinearity

Sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0$$

$$(h+1)F(x,h+1) = \sum_{r=0}^{h}(h-r+1)G(x,r)U(x,h-r+1)$$

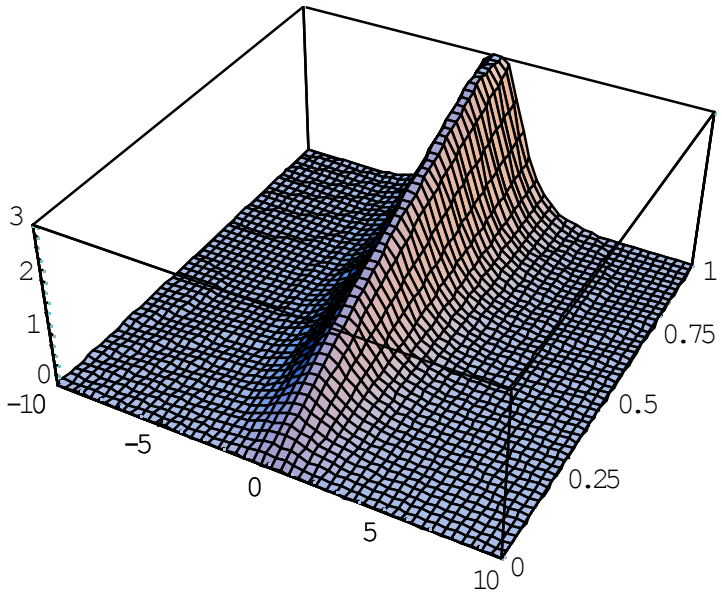
$$(h+1)G(x,h+1) = -\sum_{r=0}^{h}(h-r+1)F(x,r)U(x,h-r+1)$$

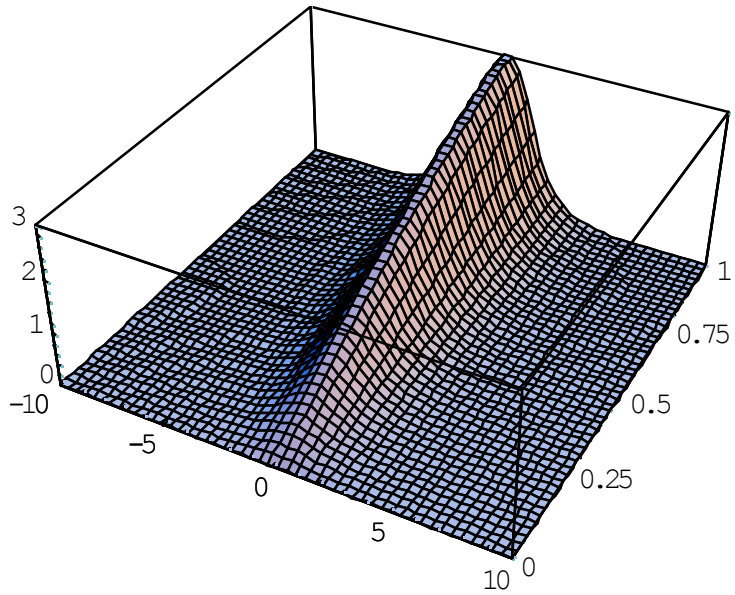
$$F(x,0) = \sin(U(x,0)), G(x,0) = \cos(U(x,0))$$

Examples of Projected DTM

Example 1

$$u(x,0) = 0, u_t(x,0) = 4\sec h(x)$$





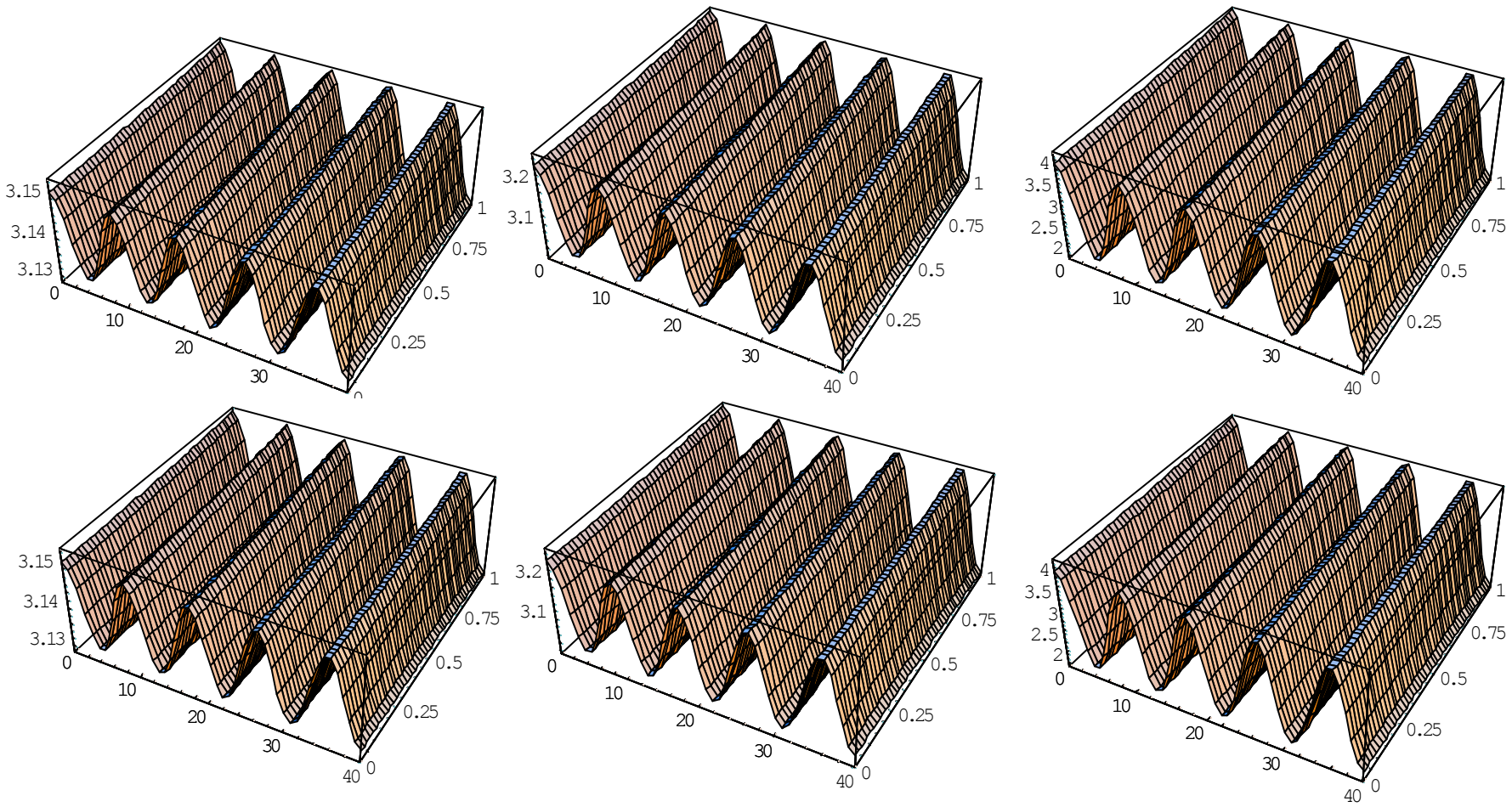
Approximate: s_5

Exact: $u(x,t) = 4\tan^{-1}[t\sec h(x)]$

Examples of Projected DTM

Example 2

$$u(x,0)=0, u_t(x,0)=\pi+\varepsilon\cos(\mu x), \mu=\sqrt{2}/2$$



DTM(Top) & VIM(Bottom) : $s_2, \varepsilon = 0.01/0.1/1.0$

Difficulties and Boundary DTM

$$y''(x)+\mu y'(x)+\gamma y(x)=f(x), \qquad y(a)=\alpha, y(b)=\beta$$

$$(k+1)(k+2)Y(k+2)+\mu(k+1)Y(k+1)+\gamma Y(k)=F(k), \qquad k=0,1,\cdots,n$$

$$\begin{bmatrix} \gamma & \mu & 2 & & \\ & \gamma & 2\mu & 6 & \\ & & \vdots & \vdots & \\ & & \gamma & (n+1)\mu & (n+1)(n+2) \end{bmatrix} \begin{bmatrix} Y[0] \\ Y[1] \\ \\ Y[n] \end{bmatrix} = \begin{bmatrix} F[0] \\ F[1] \\ \\ F[n] \end{bmatrix}$$

- (n-1) equations, n+1 unknowns
- Y[0] is known from boundary
- $y(x) \approx \sum_{k=0}^{n} Y(k)(x-x_0)^k \equiv s_n(x) \Rightarrow y(b)=s_n(b)$

Multistage Boundary DTM

•Condition at each interface

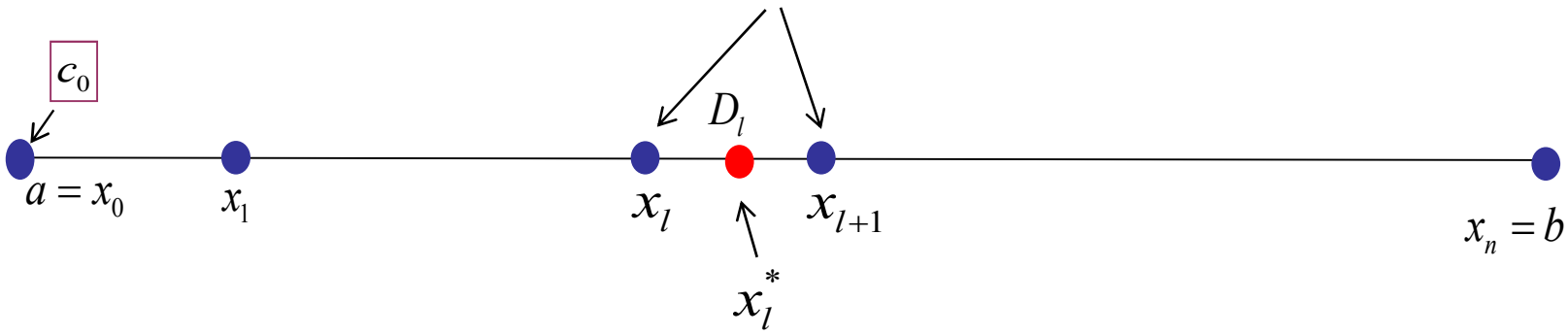
1) Continuous condition:

$$y_{l-1}(x_l) = y_l(x_l) \Rightarrow \sum_{k=0}^{n} Y_{l-1}(k)(x_l - x_{l-1}^*)^k = \sum_{k=0}^{n} Y_l(k)(x_l - x_l^*)^k$$

2) Derivative continuous condition:

$$y_{l-1}'(x_l) = y_l'(x_l) \Rightarrow \sum_{k=1}^{n} kY_{l-1}(k)(x_l - x_{l-1}^*)^{k-1} = \sum_{k=1}^{n} kY_l(k)(x_l - x_l^*)^{k-1}$$

•Interface Conditions

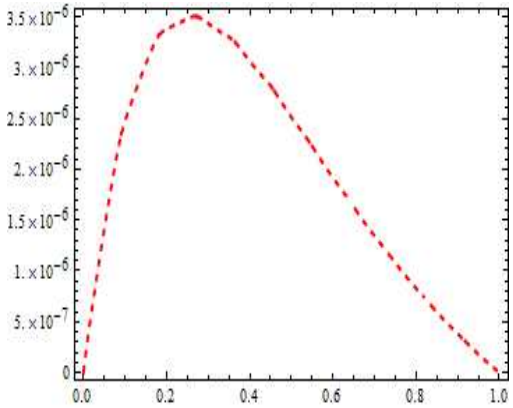


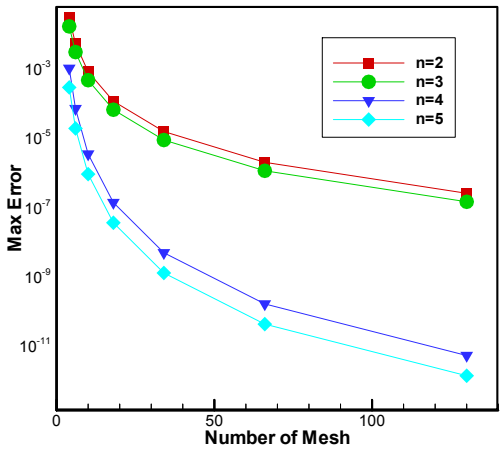
Multistage BDTM: Numerical Examples

(1) $y''+3y'-2y=2x+3 \qquad y(0)=2,\&y(1)=1$

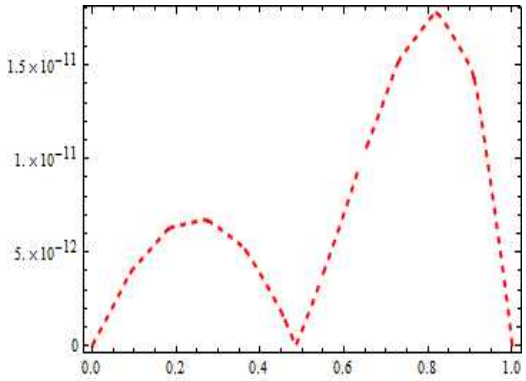
(2) $y''-2y'+2y=e^{2x}\cdot\sin x \qquad y(0)=-0.4,\&y(1)=-0.35339886$

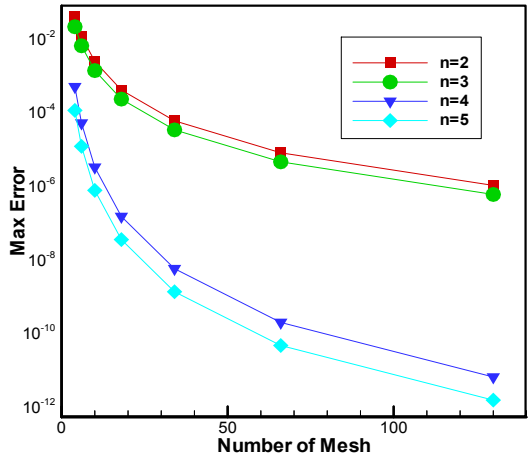
Ex1)





Ex2)





N=7, m=10

Multistage Boundary DTM: Nonlinear problems

TPBV $y'' = f(x, y', y) \quad y(a) = \alpha, y(b) = \beta$

Shooting Method

$$v'' = f(x, v, v'),\ v(a) = \alpha, v'(a) = s_0$$

Newton's Method

$$v(b, s) - \beta = 0$$

$$s_{n+1} = s_n - \frac{v(b, s_n) - \beta}{\frac{\partial}{\partial s}[v(b, s_n)]}$$

Let $\frac{\mathrm{d}v}{\mathrm{d}s} = w \Rightarrow w'' = \frac{\partial f}{\partial v}\left(x, v(x,s), v'(x,s)\right)w + \frac{\partial f}{\partial v'}\left(x, v(x,s), v'(x,s)\right)w'$

Then

$$s_{n+1} = s_n - \frac{v(b, s_n) - \beta}{w(b, s_n)}$$

Apply MDTM $w'' = f_v w + f_{v'} w', w(a) = 0, w(a) = 1$

$$Y[k+2] = \frac{1}{(k+2)(k+1)} F[k] \quad Y[0] = \alpha,\ Y[1] = s_n$$

$$W[k+2] = \frac{1}{(k+2)(k+1)} \sum_{r=0}^{k} G_1[r] W[k-r] + \sum_{r=0}^{k} G_2[r](k-r+1) W[k-r+1]$$

Multistage BDTM: Numerical Examples

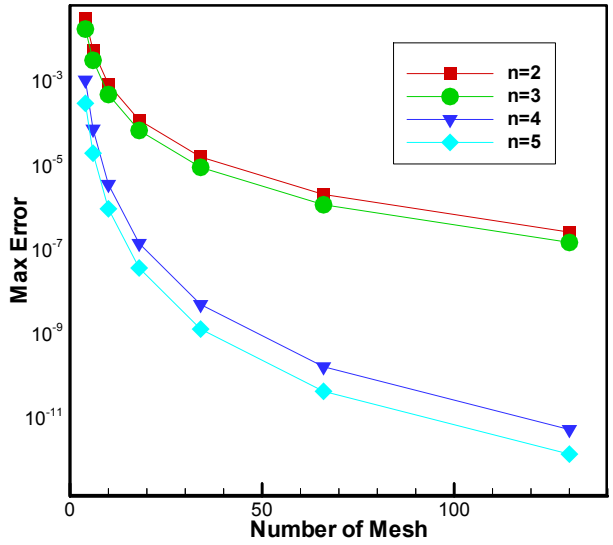
(1)

$$y'' = \frac{(32 + 2x^3 - yy')}{8}, \; y(1) = 17, \; y(3) = \frac{43}{3}$$

$$y(x) = x^2 + \frac{16}{x}$$

Ex1)

X	RK4	MDTM 5	MDTM 7
1.4	1.0E-6	2.77E-4	2.10E-6
1.8	1.0E-6	1.44E-4	1.18E-6
2.2	5.0E-6	7.34E-5	5.63E-7
2.6	9.0E-6	2.71E-5	1.96E-7



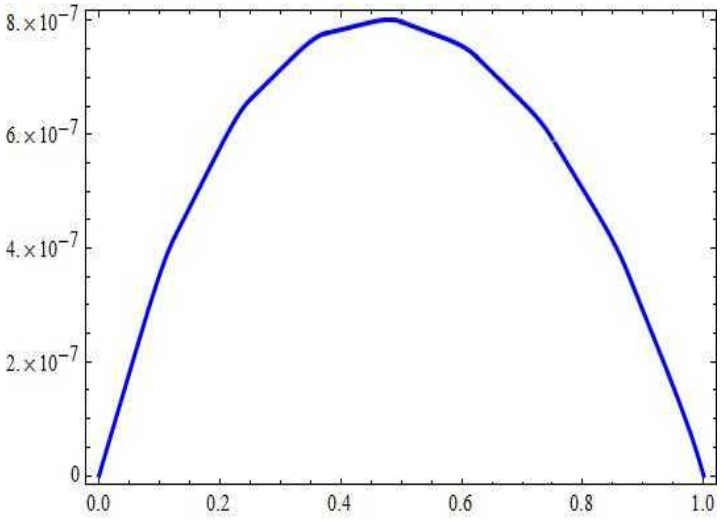
s_0=1.0 m=20(h=0.05)

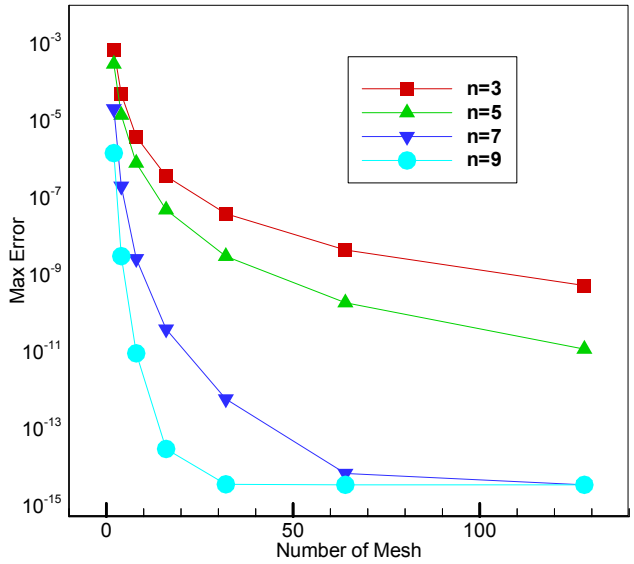
Multistage BDTM: Numerical Examples

(2) $y'' = \exp[y],\ y(0) = y(1) = 0$

$$y(x) = c\ln[c/\cos[c(x-1/2)/2]] - \ln[2], c^2 = 1 + \cos[c/2]$$

n=5,m=8





Fractional Differential Equations

- The theory of derivative of non-integer order: Leibniz's note in Sep.20 1695
- Liouville, Gruneald, Letnikov and Riemann
- For three centries the theory of fractional derivative developed as pure mathematical field
- In the last few decades, fractional derivatives are useful for description of various real materials
- Fractional order modeling in many elds(control, viscoelasticity, diusion)
- Numerical methods for solving FDE ???

Fractional Differential Equations

$$f'(t) = \lim_{h->0} \frac{f(t) - f(t-h)}{h}$$

$$f''(t) = \lim_{h->0} \frac{f'(t) - f'(t-h)}{h} = \lim_{h->0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}$$

$$f^{(n)}(t) = \lim_{h->0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t-rh)$$

Grunwald-Letnikov Fractional Derivative

$$f^{(\alpha)}(t) = \lim_{h->0} \frac{1}{h^\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t-rh)$$

Fractional Differential Equations

Riemann-Liouville Fractional Derivative

$$J_a^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-\tau)^{\mu-1} f(\tau) d\tau \qquad m-1 < \alpha \le m$$

$$D_a^{\alpha} f(t) = D^m J_a^{m-\alpha} f(t) = \left(\frac{d}{dt}\right)^m \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau$$

Properties

$$(J_a^{\alpha} J_a^{\beta} f)(t) = (J_a^{\beta} J_a^{\alpha} f)(t) = (J_a^{\alpha+\beta} f)(t)$$

$$J_a^{\alpha} (t-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}$$

Fractional Differential Equations

Caputo Fractional Derivative

$$D_a^{\alpha} f(t) = J_a^{\alpha} D^m f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} \left(\frac{d}{dt} \right)^m f(\tau) d\tau$$

$$J_a^{\alpha} D_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(t-a)^k}{k!}$$

Generalized Taylor's Formula

Generalized Taylor 's Formula

Suppose that $(D_a^\alpha)^k f(t) \in C[a,b], k = 0,1,\cdots,n+1, 0 < \alpha \le 1$

Then we have

$$f(t) = \sum_{i=0}^{n} ((D_a^\alpha)^i f)(a) \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} + \frac{((D_a^\alpha)^{n+1} f)(\eta)}{\Gamma((n+1)\alpha+1)} (t-a)^{(n+1)\alpha}$$

with $0 \le \eta \le t, \forall t \in (a,b]$

Fractional DTM

$F_\alpha(k)$: **Fractional differential transform of f(t) defined by**

$$F_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)}((D_{t_0}^{\alpha})^k f)(t_0) \Rightarrow f(t) = \sum_{k=0}^{\infty} F_\alpha(k)(t-t_0)^{\alpha k}$$

Properties

$$f(t) = g(t)h(t) \Rightarrow F_\alpha(k) = \sum_{r=0}^{k} G_\alpha(r)H_\alpha(k-r)$$

$$f(t) = D_{t_0}^{\alpha} g(t) \Rightarrow F_\alpha(k) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} G_\alpha(k+1)$$

$$f(t) = D_{t_0}^{\beta} g(t), m-1 < \beta \le m \Rightarrow F_\alpha(k) = \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} G_\alpha(k+\frac{\beta}{\alpha})$$

Fractional DTM and Examples

Fractioanal Damped Van der Pol Equations

$$\ddot{x}(t) + \mu(x^2 - 1)D^{\alpha}x(t) + x(t) = f(t)$$

Fractioanal DTM

$$\frac{\Gamma(\alpha k + \beta + 1)}{\Gamma(\alpha k + 1)} X(k + \frac{\beta}{\alpha}) + \mu \sum_{i=0}^{k} \frac{\Gamma(\alpha(k-i+1)+1)}{\Gamma(\alpha(k-i)+1)} W(i) X(k+1-i)$$

$$+ X(k) = F(k)$$

$$W(i) = \sum_{i=0}^{k} X(i)X(k-i) + \delta(i)$$

Fractional DTM and Examples

$$f(t) = a\sin(\varpi t), x(0) = x'(0) = 0$$

Table: Comparisons between the FDM solution and GDTM solution for $\alpha = 0.25, a = 1.31, \omega = 0.5,$

t	FDM	$GDTM12$	$GDTM19$	$GDTM20$
0.1	0.00010925	0.00010916	0.00010932	0.00010925
0.2	0.00087521	0.00087333	0.00087739	0.00087521
0.3	0.00295882	0.00294750	0.00297534	0.00295876
0.4	0.00702624	0.00698666	0.00709586	0.00702599
0.5	0.01374864	0.01364583	0.01396098	0.01374776
0.6	0.02380110	0.02358000	0.02432924	0.02379869
0.7	0.03786119	0.03744417	0.03900234	0.03785561
0.8	0.05660685	0.05589333	0.05883151	0.05659578
0.9	0.08071369	0.07958250	0.08472356	0.08069469
1.0	0.11085041	0.10916667	0.11764676	0.11082385

Fractional DTM and Examples

Table: Comparisons between the FDM solution and GDTM solution for $\alpha = 0.75$,Example2

t	FDM	$GDTM12$	$GDTM19$	$GDTM20$
0.1	0.00011017	0.00010916	0.00011023	0.00011017
0.2	0.00089176	0.00087333	0.00089364	0.00089145
0.3	0.00304728	0.00294750	0.00306124	0.00304466
0.4	0.00731498	0.00698666	0.00737295	0.00730309
0.5	0.01446811	0.01364583	0.01464302	0.01442981
0.6	0.02531253	0.02358000	0.02574420	0.02521365
0.7	0.04068318	0.03744417	0.04161115	0.04046442
0.8	0.06143916	0.05589333	0.06324334	0.06100761
0.9	0.08845674	0.07958250	0.09170763	0.08767876
1.0	0.12261928	0.10916667	0.12814052	0.12131760

Fractional DTM and Examples

Fractioanal Differential Equations

$$D^{\alpha}(x-x_0)(t)=\beta x(t)+f(t), x(0)=x_0$$

$$f(t)=4\exp(t)+\frac{t^{-\alpha}}{\Gamma(1-\alpha)}(F_{1,1}(1;1-\alpha,t)-4), x(0)=1$$

$$x(t)=\exp(t)$$

Fractional DTM and Examples

Table: Exact solution is $x(t) = \exp(t)$

k	$X[k], \alpha = 0.25$	$X[k], \alpha = 0.5$	$X[k], \alpha = 0.75$
1	0	0	0
2	0	1	0
3	0	0	0
4	1	$\frac{1}{2}$	1
5	0	0	0
6	0	$\frac{1}{6}$	0
7	0	0	0
8	$\frac{1}{2}$	$\frac{1}{24}$	$\frac{1}{2}$
9	0	0	0
10	0	$\frac{1}{120}$	0
11	0	0	0
12	$\frac{1}{6}$	$\frac{1}{720}$	$\frac{1}{6}$

Conclusion

- Applying the DTM to many nonlinear problems.
- Accurate approximate solutions with simple computation
- Overcome difficulties arising conventional DTM for solving problems by introducing enhanced type DTM such as MDTM, PDTM, MBDTM
- Fractional DTM can handle for solving Fractional problems
- Difficulties: Treatment of boundary in multidimensional prloblem
- Current and Future work: Multistage Fractional DTM, Integral type equations, Error Analysis

Joint work with Hyunju Kim(U North Carolina, Charlotte), Kyungho Cho(Undergraduate Student) and supported by NRF, URP

Thank you for your attention