Unique Origin Unique Future

Mathematical Modeling Lecture

# Predator-prey model
## – an example of the mathematical modeling

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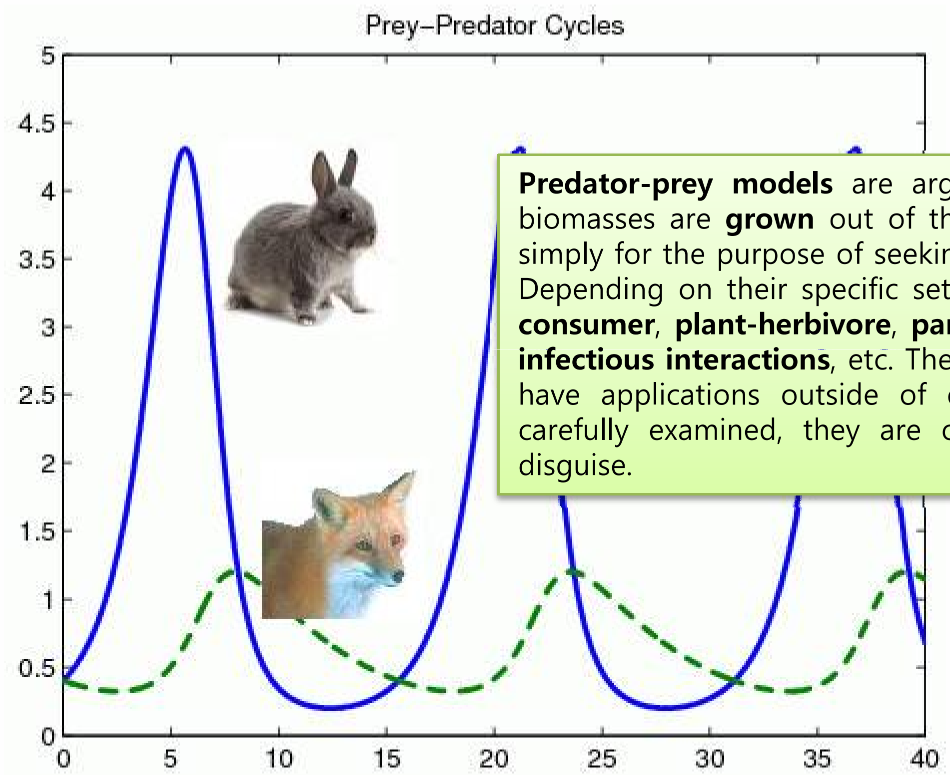
## Predator-prey model




**Predator-prey models** are argubly the building blocks of the bio- and ecosystems as biomasses are **grown** out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their specific settings of applications, they can take the forms of **resource-consumer**, **plant-herbivore**, **parasite-host**, tumor cells (virus)-immune system, **susceptible-infectious interactions**, etc. They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise.

## A General Predator-Prey Model

Consider two populations whose sizes at a reference time t are denoted by x(t), y(t), respectively. The functions x and y might denote population numbers or concentrations (number per area) or some other scaled measure of the populations sizes, but are taken to be continuous functions

$$\frac{dx}{dt} = xf(x, y)$$

$$\frac{dy}{dt} = yg(x, y)$$

Reference time : t

Population(Prey) : x(t)

Population(Predators) : y(t)

respective per capita growth rates

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$$\frac{dx}{dt} = xf(x, y)$$

$$\frac{dy}{dt} = yg(x, y)$$

$$\frac{df(x,y)}{dy} < 0 \qquad \frac{dg(x,y)}{dx} > 0$$


This general model is often called Kolmogorov's predator-prey model (Freedman 1980, Brauer and Castillo-Chavez 2000).

## Lotka-Volterra Model

In 1926, the famous Italian mathematician Vito Volterra proposed a differential equation model to explain the observed increase in predator fish (and corresponding decrease in prey fish) in the Adriatic Sea during World War I. At the same time in the United States, the equations studied by Volterra were derived independently by Alfred Lotka (1925) to describe a hypothetical chemical reaction in which the chemical concentrations oscillate. The Lotka-Volterra model is the simplest model of predator-prey interactions. It is based on linear per capita growth rates, which are written as

$$f(x,y) = b - py$$

$$g(x,y) = rx - d$$

$$\frac{dx}{dt} = (b - py)x$$

$$\frac{dy}{dx} = (rx - d)y$$

- $b$: the growth rate of species x
- $p$: the impact of predation on x
- $d$: the death (or emigration) rate of species y
- $r$: the net rate of growth (or immigration) of the predator population

This system is referred to as the **Lotka-Volterra model**: it represents one of the earliest models in mathematical ecology.

## Lotka-Volterra Model with Excel

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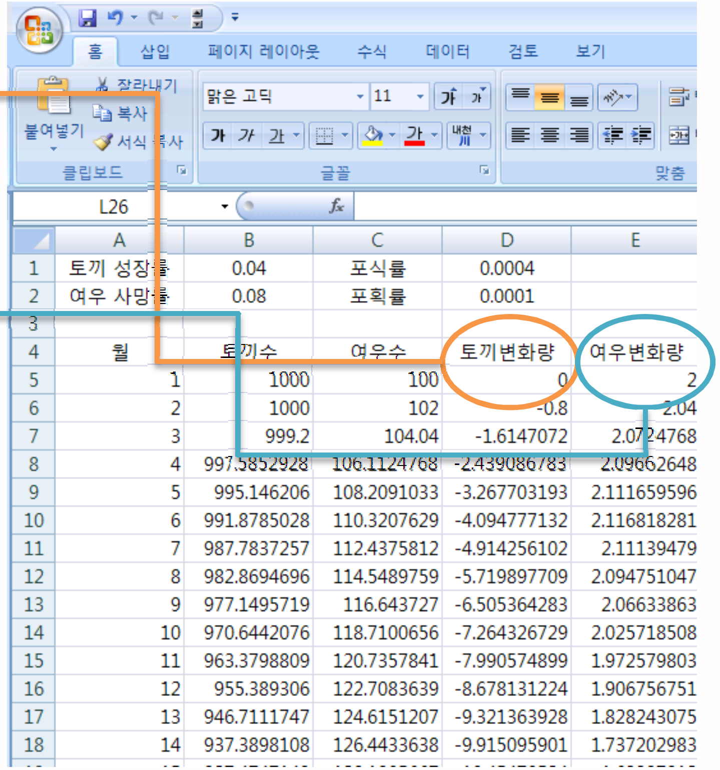
$$\frac{dx}{dt} = (b - py)x$$

=B5*($B$1-$D$1*C5)

$$\frac{dy}{dx} = (rx - d)y$$

=C5*(-$B$2+$D$2*B5)

b (the growth rate of species x) : 0.04
p (the impact of predation on x) : 0.0004
d (the death (or emigration) rate of species y) : 0.08
r (the net rate of growth (or immigration) of the predator population) : 0.0001



| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | 토끼 성장률 | 0.04 | 포식률 | 0.0004 | |
| 2 | 여우 사망률 | 0.08 | 포획률 | 0.0001 | |
| 3 | | | | | |
| 4 | 월 | 토끼수 | 여우수 | 토끼변화량 | 여우변화량 |
| 5 | 1 | 1000 | 100 | 0 | 2 |
| 6 | 2 | 1000 | 102 | -0.8 | 2.04 |
| 7 | 3 | 999.2 | 104.04 | -1.6147072 | 2.0724768 |
| 8 | 4 | 997.5852928 | 106.1124768 | -2.439086783 | 2.09662648 |
| 9 | 5 | 995.146206 | 108.2091033 | -3.267703193 | 2.111659596 |
| 10 | 6 | 991.8785028 | 110.3207629 | -4.094777132 | 2.116818281 |
| 11 | 7 | 987.7837257 | 112.4375812 | -4.914256102 | 2.11139479 |
| 12 | 8 | 982.8694696 | 114.5489759 | -5.719897709 | 2.094751047 |
| 13 | 9 | 977.1495719 | 116.643727 | -6.505364283 | 2.06633863 |
| 14 | 10 | 970.6442076 | 118.7100656 | -7.264326729 | 2.025718508 |
| 15 | 11 | 963.3798809 | 120.7357841 | -7.990574899 | 1.972579803 |
| 16 | 12 | 955.389306 | 122.7083639 | -8.678131224 | 1.906756751 |
| 17 | 13 | 946.7111747 | 124.6151207 | -9.321363928 | 1.828243075 |
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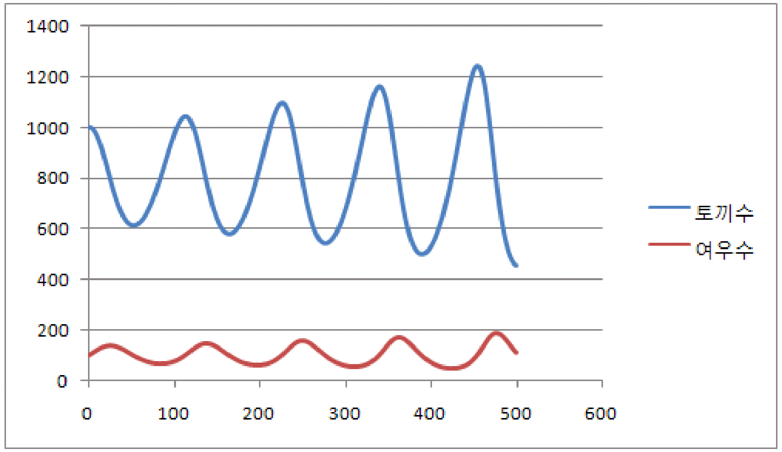


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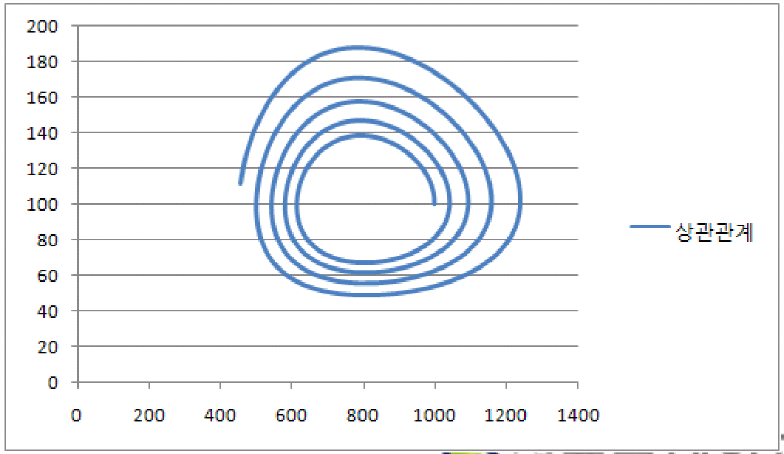

## Lotka-Volterra Model : Bifurcation

**Bifurcation theory** is the mathematical study of changes in the qualitative or topological structure of a given family. Examples of such families are the integral curves of a family of vector fields or, the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour. Bifurcations occur in both continuous systems (described by ODEs, DDEs or PDEs), and discrete systems (described by maps). :

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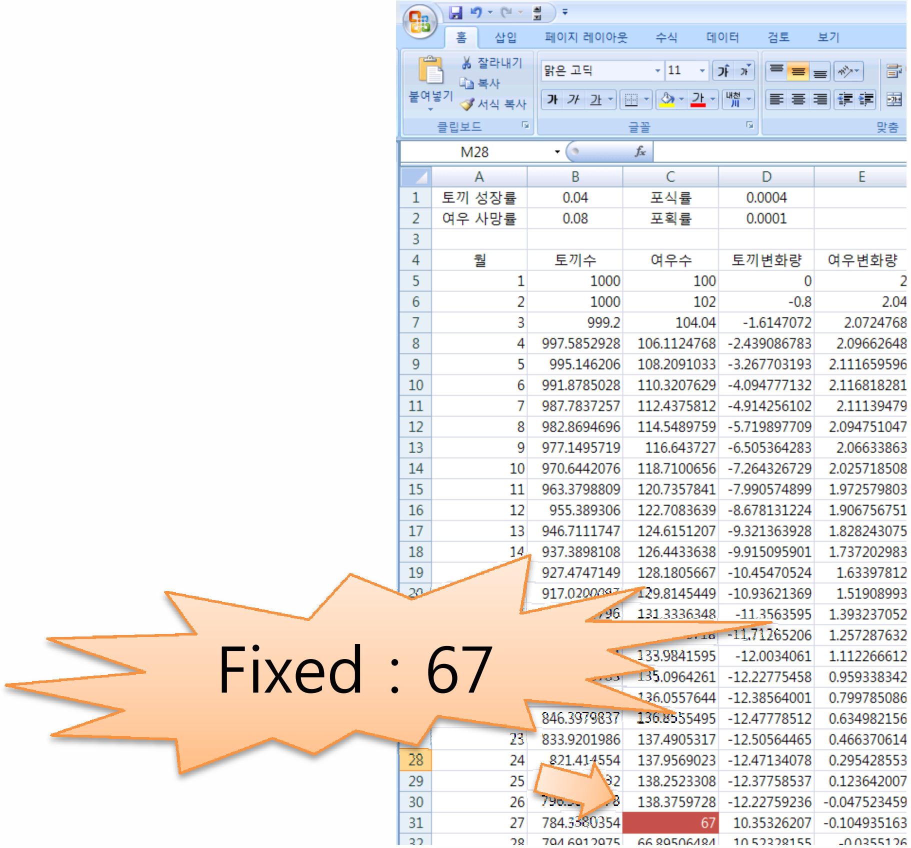
27th-month : Hunting predators (138 → 67)

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| 18 | 14 | 937.3898108 | 126.4433638 | -9.915095901 | 1.737202983 |
| 19 | [illegible] | 927.4747149 | 128.1805667 | -10.45470524 | 1.63397812 |
| 20 | [illegible] | 917.0200097 | 129.8145449 | -10.93621369 | 1.51908993 |
| [illegible] | [illegible] | [illegible] | 131.3336348 | -11.3563595 | 1.393237052 |
| [illegible] | [illegible] | [illegible] | [illegible] | -11.71265206 | 1.257287632 |
| [illegible] | [illegible] | [illegible] | 133.9841595 | -12.0034061 | 1.112266612 |
| [illegible] | [illegible] | [illegible] | 135.0964261 | -12.22775458 | 0.959338342 |
| [illegible] | [illegible] | [illegible] | 136.0557644 | -12.38564001 | 0.799785086 |
| [illegible] | [illegible] | 846.3979837 | 136.8555495 | -12.47778512 | 0.634982156 |
| [illegible] | 23 | 833.9201986 | 137.4905317 | -12.50564465 | 0.466370614 |
| 28 | 24 | 821.414554 | 137.9569023 | -12.47134078 | 0.295428553 |
| 29 | 25 | [illegible] | 138.2523308 | -12.37758537 | 0.123642007 |
| 30 | 26 | [illegible] | 138.3759728 | -12.22759236 | -0.047523459 |
| 31 | 27 | 784.3380354 | 67 | 10.35326207 | -0.104935163 |
| 32 | 28 | 794.6912975 | 66.89506484 | 10.52328155 | -0.0355126 |

## Lotka-Volterra Model : Bifurcation

In particular, fixed points can be created or destroyed, or their stability can change. These qualitative chnages in the dynamics are called **bifurcations** and the parameter values at which they occur are called **bifurcation points**.

$$\frac{dx}{dt} = (b - py)x$$

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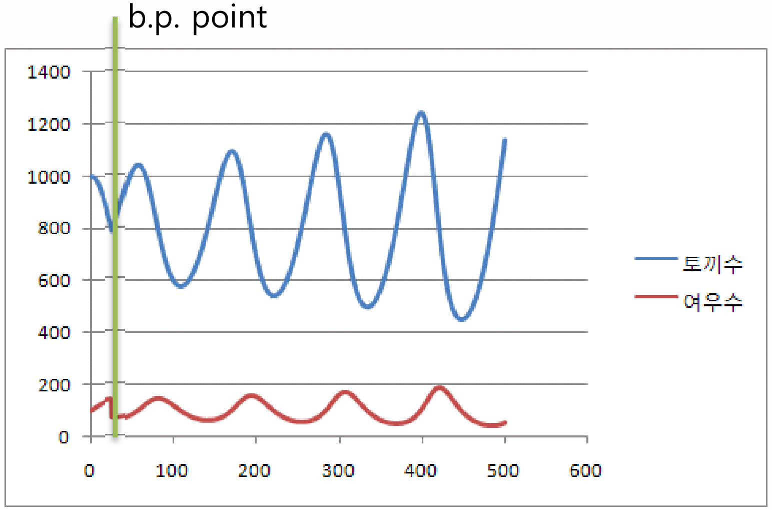


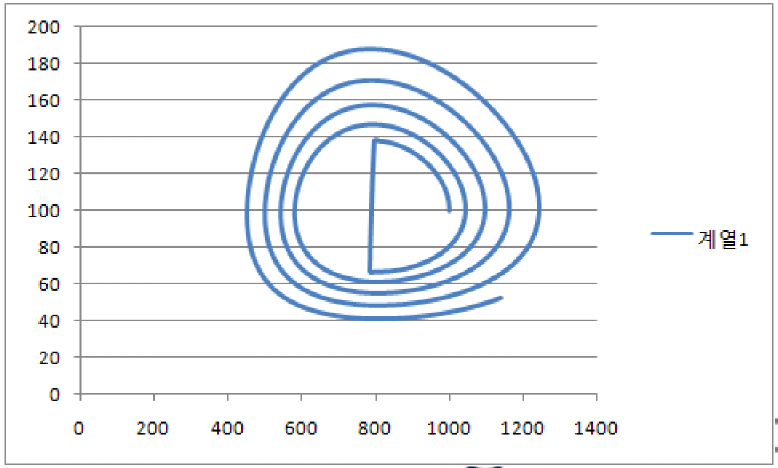


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27th-month : Hunting predators (138 → 67)

## Lotka-Volterra Model : Equilibrium

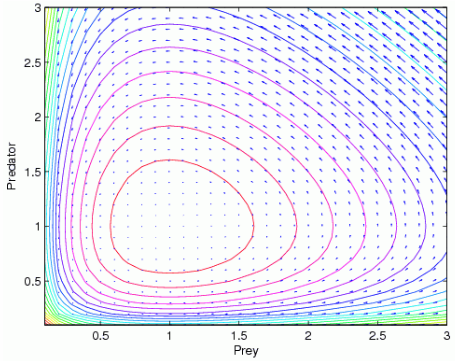
**An equilibrium (or equilibrium point)** of a dynamical system generated by an autonomous system of ordinary differential equations (ODEs) is a solution that does not change with time. For example, each motionless pendulum position in Fig.1 corresponds to an equilibrium of the corresponding equations of motion, one is stable, the other one is not. Geometrically, equilibria are points in the system's phase space.

$$\frac{dx}{dt} = (b - py)x$$

$$\frac{dy}{dx} = (rx - d)y$$

$$C = b\ln(y(t)) - py(t) - rx(t) + d\ln(x(t))$$

$$z = b\ln(y) - py - rx + d\ln(x)$$




$$z = \ln(y) - y - x + \ln(x)$$

http://math1.skku.ac.kr/home/pub/399/

## Lotka-Volterra Model : Application to other models

The model above has been derived independently in the following fields:

$$\frac{dx}{dt} = (b - py)x \qquad \frac{dy}{dx} = (rx - d)y$$

X (susceptible) → Y (infective)

**Epidemics (Kermak and McKendrick 1927, 1932, 1933)**
b=0

X (chemical radicals) → Y (chemical radicals)

**Combustion theory (Semenov 1935)**
$H_2O_2$ combustion

X (populace) → Y (predatory institution)

**Economics (Galbraith 2006)**

## Other Predator-Prey Models

General Predator-Prey Model

Lotka-Volterra Model

**Kermack-McKendrick Model**

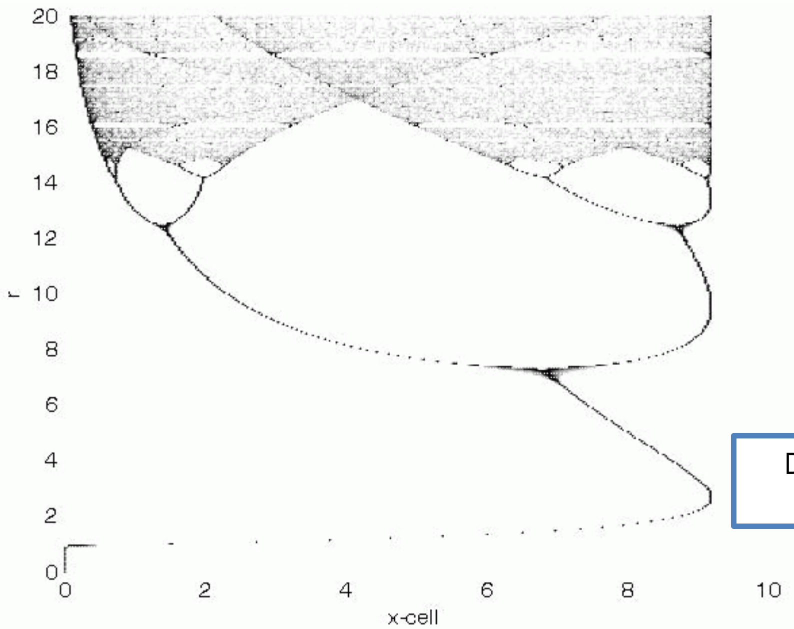
**Jacob-Monod Model**

Biochemical Process

**Predation with Time Delays: Chaos in Ricker's Reproduction Equation**

Malthus's model

Chaotic dynamics




Dynamics of Ricker's population.

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