Fall 2015, LA Midterm Exam Sol. (1 hour In class Exam)							Sign
Course	Linear Algebra	GEDB		Prof.			
Major		Year 학년		Student No. 학번		Name	

※ Notice

1. Fillout the above boxes before you start this Exam. (학번, 이름 등을 기입하고 감독자 날인)
2. Honor Code: (시험 부정행위시 해당 교과목 성적이 "F" 처리됨은 물론 징계위원회에 회부될 수 있습니다.)
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4. You may use the following <Sage codes> in your answers. (중간고사까지는 한국어 답안도 OK)

Total Score (100 pt)	
Offline Exam 85	Participation 15

```
var('a, b, c, d')                        # Define variables
eq1=3*a+3*b==12                          # Define equation1
eq2=5*a+2*b==13                          # Define equation2
solve([eq1, eq2], a,b)                   # Solve eq's
A=matrix(QQ, 3, 3, [3, 0, 0, 0, 0, 2, 0, 3, 4]);   # Matrix
x=vector([3, 1, 2])                      # Define vector x
A.augment(x)                             # [A: x]
A.echelon_form()                         # Find RREF
A.inverse()                              # Find inverse
A.det()                                  # Find determinant
A.adjoint()                              # Find adjoint matrix
A.charpoly()                             # Find charct. ploy
A.eigenvalues()                          # Find eigenvalues
A.eigenvectors_right()                   # Find eigenvectors
A.rank()                                 # Find rank of A
A.right_nullity()                        # Find nullity of A

A=random_matrix(QQ,7,7)   # random matrix of size 7 over Q
bool( A== B)              # Are A and B same?
P,L,U=A.LU()              # LU (P: Permutation M. / L, U

var('x, y')                              # Define variables
f = 7*x^2 + 4*x*y + 4*y^2-23             # Define a function
implicit_plot( f, (x, -10, 10), (y, -10, 10))   # implicit Plot
plot3d(y^2+1-x^3-x, (x, -pi, pi), (y, -pi, pi))  # 3D Plot

var('t')                                 # Define variables
x=2+2*t                                  # Define a parametric eq.
y=-3*t-2
parametric_plot((x,y), (t, -10, 10), rgbcolor='red')  # Plot
[G,mu]=A.gram_schmidt()       # G-S
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B #
A.jordan_form()     # Jordan Canonical Form of A
```

<Sample Sage Linear Algebra codes>

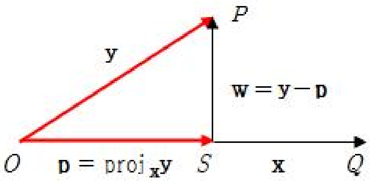
I. (1pt x 20= 20pt) True(T) or False(F).

1. (T) For each $\mathbf{y}$ and each subspace W of $\mathbb{R}^n$, the vector $\mathbf{y}-\mathrm{proj}_{\mathbf{W}}\mathbf{y}$ is orthogonal to W.
2. (F) A system of six linear equations with 3 unknowns cannot have more than 1 solution.
3. (T) A linear system of the form $A\mathbf{x}=\mathbf{0}$ containing eight equations and ten unknowns has infinitely many solutions.
4. (T) Not every linear independent set in $\mathbb{R}^n$ is an orthogonal set.
5. (T) Every linear system of the form $A\mathbf{x}=\mathbf{0}$ has at least 1 solution.
6. (T) A given matrix can be written uniquely as a sum of a symmetric matrix and a skew-symmetric matrix.
7. (F) Any subspace of $\mathbb{R}^2$ is either a line through the origin or $\mathbb{R}^2$.
8. (T) $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1-2x_3=0\}$ is a subspace of $\mathbb{R}^3$
9. (T) For any $n\times n$ matrix A with $n>1$, $\det(\mathrm{adj}\,A)=\det(A)^{n-1}$.
10. (T) Let A be an $n\times n$ invertible matrix, then the inverse matrix of A is $A^{-1}=\frac{1}{|A|}\mathrm{adj}\,A$.
11. (T) For a set of natural numbers $S=\{1,2,\ldots,n\}$, permutation is a one to one function from S to S.
12. (T) The determinant of matrix $A=[a_{ij}]$ in M_n, is defined as $\det(A)=\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$.
13. (T) For any two $n\times n$ matrices A and B, $\det(AB)=\det(B)\det(A)$
14. (T) A matrix with all orthonormal columns is an orthogonal matrix.
15. (T) If the columns of an $m\times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x}\mapsto A\mathbf{x}$ preserves length.
16. (T) For any invertible lower triangular matrix A, A^{-1} is a lower triangular matrix.
17. (F) There is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ whose image is $\mathbb{R}^3$.
18. (F) For a transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, if $T(\mathbf{u})=T(\mathbf{v}) \Rightarrow \mathbf{u}=\mathbf{v}$, then it is called onto.
19. (F) For a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, $\mathrm{Im}\,T$ is a subspace of $\mathbb{R}^n$.
20. (T) If a LT $T:\mathbb{R}^n\to\mathbb{R}^m$ is one-to-one and onto, then $n=m$ and T is called an isomorphism.

II. (2pt x 5 = 10pt) State or Define (Choose 5: Mark only 5 and Fill the boxes and/or state).

1. [$\mathrm{proj}_{\mathbf{x}}\mathbf{y}$] The (vector) projection of $\mathbf{y}$ onto $\mathbf{x}$ and is denoted by $\mathrm{proj}_{\mathbf{x}}\mathbf{y}$.

Here, the vector $\mathbf{w} = \overrightarrow{SP} = \mathbf{y} - \mathbf{p}$ is called the component of $\mathbf{y}$ orthogonal to $\mathbf{x}$. Therefore, $\mathbf{y}$ can be written as $\mathbf{y} = \mathbf{p} + \mathbf{w}$. For vectors $\mathbf{x}\,(\neq 0)$, $\mathbf{y}$ in $\mathbb{R}^3$, we have the following:



$$\mathrm{proj}_{\mathbf{x}}\mathbf{y} = t\,\mathbf{x} \quad \text{where} \quad t = \frac{\mathbf{y}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}}.$$

2. [cofactor expansion] Let A be an $n \times n$ matrix. For any i, j $(1 \leq i, j \leq n)$ the following holds.

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ (cofactor expansion along the ith row)

$|A| =$ $a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ (cofactor expansion along the jth column)

3. [eigenspace] Let A be an $n\times n$ matrix. For a nonzero vector $\mathbf{x}\in\mathbb{R}^n$, if there exist a scalar λ which satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then λ is called an eigenvalue of A, and $\mathbf{x}$ is called an eigenvector of A corresponding to λ.

Define an eigenspace of A corresponding to λ =

the solution space of the system of linear equations = $\{\mathbf{x} \in \mathbb{R}^n \mid (\lambda I_n - A)\mathbf{x} = 0\}$.

4. [kernel] Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$\ker T = \{\mathbf{v}\in\mathbb{R}^n \mid T(\mathbf{v}) = 0 \in \mathbb{R}^m\}$$

•

※ State the following concepts :

5. [Span of S]

the span of S is defined as the set of all linear combinations of elements of S.

6. [Linearly independent, linearly dependent]

$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent : $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0$

Otherwise, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly dependent.

7. [Cramer's Rule]

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \qquad &\;\; \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

let A be a coefficient matrix, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Then the system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$. If $|A| \neq 0$, the system of linear equations has a unique solution as follows:

$$x_1 = \frac{|A_1|}{|A|},\; x_2 = \frac{|A_2|}{|A|},\; \ldots,\; x_n = \frac{|A_n|}{|A|}$$

where $A_j\ (j = 1, 2, \cdots, n)$ denotes the matrix A with the jth column replaced by the vector $\mathbf{b}$.

III. (4pt x 7 = 28pts) Find or Explain (Fill the boxes) :

1. Find the distance D from the point $P(3,-1,2)$ to the plane $x+3y-2z-6=0$.

Sol $\mathbf{p}=\mathrm{proj}_{\mathbf{n}}\mathbf{v}=t\mathbf{n}=\dfrac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}$.

Here, $\mathbf{n}=(1,3,-2)$, $\mathbf{v}=\overrightarrow{OP_0}-\overrightarrow{OP_1}=\mathbf{x}-\mathbf{x}_1=(3,-1,2)-(x_1,y_1,z_1)$ where $x_1+3y_1-2z_1-6=0$, so

$$\mathbf{p}=\mathrm{proj}_{\mathbf{n}}\mathbf{v}=\frac{(3-x_1,-1-y_1,2-z_1)\cdot(1,3,-2)}{1^2+3^2+(-2)^2}(1,3,-2)$$

$$=\frac{-x_1-3y_1+2z_1-4}{14}(1,3,-2)=\frac{-6-4}{14}(1,3,-2)$$

$$=-\frac{5}{7}(1,3,-2)=\left(-\frac{5}{7},-\frac{15}{7},\frac{10}{7}\right).$$

$$D=\|\mathrm{proj}_{\mathbf{n}}\mathbf{v}\|=\sqrt{\left(-\frac{5}{7}\right)^2+\left(-\frac{15}{7}\right)^2+\left(\frac{10}{7}\right)^2}=\frac{5\sqrt{14}}{7}$$ □

Sage Copy the following code into http://sage.skku.edu to practice.

```
n=vector([1, 3, -2])
v=vector([3, -1, 2]);d=-6
vn=v.inner_product(n)
nn=n.norm()
Distance=abs(vn+d)/nn
print Distance
```

5/7*sqrt(14) # $\dfrac{10}{\sqrt{14}}=\dfrac{5}{7}\sqrt{14}$ ■

2. Suppose that three points $(-1,7)$, $(2,15)$, $(1,3)$ pass through the parabola $y=a_0+a_1x+a_2x^2$. By plugging in these points, obtain three linear equations. Find coefficients a_0, a_1, a_2 by solving $A\mathbf{x}=\mathbf{b}$.

Sol

$$\begin{cases}a_0-a_1+a_2=7\\ a_0+2a_1+4a_2=15\\ a_0+a_1+a_2=3\end{cases}\quad (\because (-1,7),(2,15),(1,3)\text{ pass through the parabola})\ \begin{bmatrix}1&-1&1\\1&2&4\\1&1&1\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}=\begin{bmatrix}7\\15\\3\end{bmatrix},\text{ where } A=\begin{bmatrix}1&-1&1\\1&2&4\\1&1&1\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}7\\15\\3\end{bmatrix}.$$

$$[A:\mathbf{b}]=\begin{bmatrix}1&-1&1&:&7\\1&2&4&:&15\\1&1&1&:&3\end{bmatrix}\xrightarrow{R_3-R_1}\begin{bmatrix}1&-1&1&:&7\\1&2&4&:&15\\0&2&0&:&-4\end{bmatrix}\xrightarrow{\frac{1}{2}R_3}\begin{bmatrix}1&-1&1&:&7\\1&2&4&:&15\\0&1&0&:&-2\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\cdots\rightarrow\begin{bmatrix}1&-1&1&:&7\\0&1&0&:&-2\\0&0&1&:&\frac{14}{3}\end{bmatrix}\xrightarrow[R_2+R_1\rightarrow R_1]{-R_3+R_1\rightarrow R_1}\begin{bmatrix}1&0&0&:&\frac{1}{3}\\0&1&0&:&-2\\0&0&1&:&\frac{14}{3}\end{bmatrix}.$$

=> $a_0=\frac{1}{3}$, $a_1=-2$, $a_2=\frac{14}{3}$. Answer : $y=\dfrac{1}{3}-2x+\dfrac{14}{3}x^2$ ■

3. Let T_1 and T_2 are defined as follows:

$$T_1(x_1,x_2,x_3)=(4x_1,-2x_1+x_2,-x_1-3x_2),\qquad T_2(x_1,x_2,x_3)=(x_1+2x_2,-x_3,4x_1-x_3).$$

(1) Find the standard matrix for each T_1 and T_2.

(2) Find the standard matrix for each $T_2\circ T_1$ and $T_1\circ T_2$.

Sol

(1) $T_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}4\\-2\\-1\end{bmatrix}$, $T_1\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\-3\end{bmatrix}$, $T_1\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}$, $\therefore [T_1]=\begin{bmatrix}4&0&0\\-2&1&0\\-1&-3&0\end{bmatrix}$

$T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\4\end{bmatrix}$, $T_2\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\0\\0\end{bmatrix}$, $T_2\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\-1\\-1\end{bmatrix}$ $\therefore [T_2]=\begin{bmatrix}1&2&0\\0&0&-1\\4&0&-1\end{bmatrix}$

(2) $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}1&2&0\\0&0&-1\\4&0&-1\end{bmatrix}\begin{bmatrix}4&0&0\\-2&1&0\\-1&-3&0\end{bmatrix}=\begin{bmatrix}0&2&0\\1&3&0\\17&3&0\end{bmatrix}$, $[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}4&0&0\\-2&1&0\\-1&-3&0\end{bmatrix}\begin{bmatrix}1&2&0\\0&0&-1\\4&0&-1\end{bmatrix}=\begin{bmatrix}4&8&0\\-2&-4&-1\\-1&-2&3\end{bmatrix}$ ■

Sage) http://math3.skku.ac.kr/home/pub/29 (See the code and output in page 4)

```
x,y,z=var('x y z')
A(x,y,z)=(4*x,-2*x+y,-x-3*y)
a(x,y,z)=(x+2*y,-z,4*x-z)
T=linear_transformation(QQ^3, QQ^3,A)
t=linear_transformation(QQ^3, QQ^3,a)
C = T.matrix(side='right')
c = t.matrix(side='right')
print "[T1]="
print C
print "[T2]="
print c
print "[T2*T1]="
print c*C
print "[T1*T2]="
print C*c
```

```
[T1]=             [T2]=
[ 4  0  0]        [ 1  2  0]
[-2  1  0]        [ 0  0 -1]
[-1 -3  0]        [ 4  0 -1]

[T2*T1]=          [T1*T2]=
[ 0  2  0]        [ 4  8  0]
[ 1  3  0]        [-2 -4 -1]
[17  3  0]        [-1 -2  3]
```

4. Let $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ moves any $\mathbf{x} \in \mathbb{R}^2$ to a symmetric image to a line which passes through the origin and has angle $\theta = \frac{\pi}{4}$ between the line and the x-axis. Find $H_\theta(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

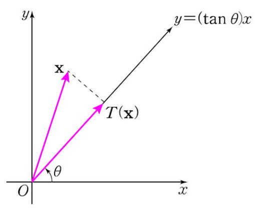
Sol The symmetric transformation H_θ which passes through the origin and has angle between the line and the x-axis is,

At $\theta = \frac{\pi}{4}$, $[H_\theta] = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\therefore\ H_\theta(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ ■

5. As shown in the picture, let us define an orthogonal projection as a linear transformation (linear operator) $P_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which maps any vector $\mathbf{x}$ in $\mathbb{R}^2$ to the orthogonal projection on a line, which passes through the origin with angle $\theta = \frac{\pi}{4}$ between the x-axis and the line. Let us denote the standard matrix corresponding to P_θ when $H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

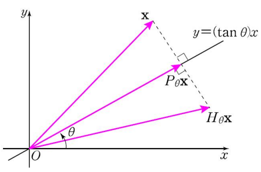
Sol $P_\theta\mathbf{x} - \mathbf{x} = \frac{1}{2}(H_\theta\mathbf{x} - \mathbf{x})$ (the same direction with a half length)



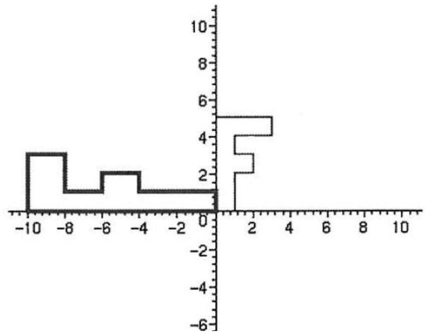
$$P_\theta\mathbf{x} = \frac{1}{2}H_\theta\mathbf{x} + \frac{1}{2}\mathbf{x} = \frac{1}{2}H_\theta\mathbf{x} + \frac{1}{2}I\mathbf{x} = \frac{1}{2}(H_\theta + I)\mathbf{x}$$

$$P_\theta = \frac{1}{2}(H_\theta + I) = \left(\begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1-\cos 2\theta) \end{bmatrix} \right.$$

$$\Rightarrow \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}_{\theta=\frac{\pi}{4}} = \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right).$$ ■



6. Find a linear transformation $T:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger **F**.):



Sol

Answer : $T(A) = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

since $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta=\frac{\pi}{2}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$. ■

7. [Invertible Matrix Theorem] Let A be an $n \times n$ matrix.

Which of the following statements is not equivalent to "the matrix A is invertible."? (Choose one)

(1) Column vectors of A are linearly independent.
(2) Row vectors of A are linearly independent.
(3) $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = \mathbf{0}$.
(4) For any $n \times 1$ vector $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.
(5) A and I_n are row equivalent.
(6) A and I_n are column equivalent.
(7) $\det(A) \neq 0$
(8) $\lambda = 0$ is an eigenvalue of A.
(9) $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_A(\mathbf{x}) = A\mathbf{x}$ is one-to-one.
(10) $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_A(\mathbf{x}) = A\mathbf{x}$ is onto.

Ans 8 . ■

IV. (3+4+5=12pt) Python/ Sage Computations.

1. (3pts) When we solve a LSE $A\mathbf{x} = \mathbf{b}$ whose augmented matrix is $B = \left[\begin{array}{cccc:c} 2 & 1 & 1 & -2 & 1 \\ 3 & -2 & 1 & -6 & -2 \\ 1 & 1 & -1 & -1 & -1 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$ and $\text{RREF}(B) = \left[\begin{array}{cccc:c} 1 & 0 & 0 & -\frac{17}{11} & 0 \\ 0 & 1 & 0 & \frac{9}{11} & 0 \\ 0 & 0 & 1 & \frac{3}{11} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

Explain why this system has no solution.

Ans The last equation in the system means $w\,0 = 1$ which is impossible when $\mathbf{x} = (x, y, z, w)$ is a solution. Therefore $A\mathbf{x} = \mathbf{b}$ has a solution set is $\varnothing$ (Empty set).■

2. **(4pts)** Consider $A\mathbf{x}=\mathbf{y}$ where $A=\begin{bmatrix}0&1&2&3\\1&2&3&4\\2&3&4&5\\3&4&5&6\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$. Similarly we have found the augmented matrix $[A:\mathbf{y}]$ and its RREF by Sage RREF($[A:\mathbf{y}]$) $=\begin{bmatrix}1&0&-1&-2&2\\0&1&2&3&-1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

(1) Find number of linear independent rows of A Ans: (2)

(2) The solution set of $A\mathbf{x}=\mathbf{y}$.

Ans: $\{(s+2t+2,\ -2s-3t-1,\ s,\ t)\,|\,s,t\in\mathbb{R}\}$ **or** $\left\{\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}+s\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}+t\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\ \middle|\ s,t\in\mathbb{R}\right\}$ ■

3. **(5pts)** Consider $A\mathbf{x}=\mathbf{y}$ where $A=\begin{bmatrix}-18&-30&-30&-36\\42&54&30&36\\-6&-6&18&0\\30&30&30&48\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$. You were asked to find

(1) Augment matrix $[A:\mathbf{y}]$ (2) RREF(A) (3) Det$A$ (4) Inverse of A (4) characteristic polynomial of A (5) all eigenvalues of A (6) all eigenvectors of A. The following is your answer. Fill out the blanks to find each.

Sol)

1) Step 1: Browse http://math3.skku.ac.kr or http://math1.skku.ac.kr/ (or http://sage.skku.edu/ or https://cloud.sagemath.com etc)
2) Step 2: Type class/your ID: (math2013 or yours) and PW : (math**** or yours)
3) Step 3: Click "New worksheet (새 워크시트)" button.
4) Step 4: Define a matrix A in the first cell in rational (QQ) field.
   A = matrix(QQ, 4, 4, [-18, -30,-30,-36,42,54,30,36,-6,-6,18,0,30,30,30,48]) and y = matrix(QQ, 4, 1, [-1, 0, 1, 2])
5) Step 5: Type a command to find augment matrix **[A: y]** A.augment(y) and evaluate
6) Step 6: Type a command to find **RREF**(A) **A.echelon_form()** and evaluate.
7) Step 7: Type a command to find **determinant of** A **A.det()** and evaluate.
8) Step 8: Type a command to find inverse of A **A.inverse()** and evaluate.
9) Step 9: Type a command to find char. polynomial of A **A.charpoly()** and evaluate.
10) Step 10: Type a command to find **eigenvalues** of A **A.eigenvalues()** and evaluate.
11) Step 11: Type a command to find **eigenvectors** of A **A.eigenvectors_right()** and evaluate.
13) Last step : Give 'print' command to see what you like to read.

Now we have some out from the Sage.

RREF(A) = Identity matrix of size 4

det(A) = 248832

inverse(A) =

```
[ 17/144    5/144    5/144    1/16]
[-11/144    1/144   -5/144   -1/16]
[   1/72     1/72     1/18       0]
[ -5/144   -5/144   -5/144    1/48]
```

characteristic polynomial of (A) = x^4 - 102*x^3 + 3528*x^2 - 50112*x + 248832

eigenvalues of A = { 48, 24, 18, 12}

eigenvectors = [(48, [(1, -1, 0, -1)], 1), (24, [(0, 1, -1, 0)], 1), (18, [(1, -1, 1, -1)], 1), (12, [(1, -1, 0, 0)], 1)]

Write what (24, [(0, 1, -1, 0)], 1) means in eigenvectors of A :

24 : eigenvalue, [(0, 1, -1, 0)] : corresponding eigenvector , 1 : algebraic multiplicity of engenvalue 24 ,

■

V. (3pt x 5 = 15pt) Explain or give a sketch of proof.

1. If $A^2 = A$, show that $(I-2A) = (I-2A)^{-1}$.

Proof Show $(I-2A)(I-2A) = I$ when $A^2 = A$

$$(I-2A)(I-2A) = I - 2A - 2A + 4A^2$$
$$= I - 4A + 4A = I \quad (\because A^2 = A)$$

$\therefore (I-2A)^{-1} = (I-2A)$ ■

2. Show AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ when A, B are invertible square matrices of order n.

Proof $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= AI_nA^{-1} = AA^{-1} = I_n.$$ ■

3. Let A and I be $n \times n$ matrices. If $A+I$ is invertible, show that $A(A+I)^{-1} = (A+I)^{-1}A$.

Proof $(A+I)A = A^2 + A = A(A+I)$

=> $(A+I)^{-1}(A+I)A(A+I)^{-1} = (A+I)^{-1}A(A+I)(A+I)^{-1}$ $\quad (\because A+I$ is invertible $)$

=> $A(A+I)^{-1} = (A+I)^{-1}A$ ■

4. Show $W_6 = \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$ is a subspace of $\mathbb{R}^3$.

Sol

Show 1) W_6 is closed under the vector addition.

2) W_6 is closed under the scalar multiplication.

$\forall$ $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (x_4, x_5, x_6) \in W$, $k \in \mathbb{R}$

1) $\mathbf{x} + \mathbf{y} = (x_1 + x_4, x_2 + x_5, x_3 + x_6) \in W_6 \quad (\because x_1 + x_4 = x_2 + x_5 = x_3 + x_6)$

2) $k\mathbf{x} = (kx_1, kx_2, kx_3) \in W_6 \quad (\because kx_1 = kx_2 = kx_3)$

Therefore, W_6 is a subspace of $\mathbb{R}^3$. ■

5. Show the following :

Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be vector spaces and $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then T is one-to-one **if and only if** $\ker T = \{0\}$.

Proof ($\Rightarrow$) As $\forall$ $\mathrm{v} \in \ker T$, $T(\mathrm{v}) = 0 = T(0)$ and T is one-to-one,

$\Rightarrow$ $\mathrm{v} = 0$ $\qquad \therefore \ker T = \{0\}$

($\Leftarrow$) $T(\mathrm{v}_1) = T(\mathrm{v}_2) \Rightarrow 0 = T(\mathrm{v}_1) - T(\mathrm{v}_2) = T(\mathrm{v}_1 - \mathrm{v}_2)$

$\Rightarrow \mathrm{v}_1 - \mathrm{v}_2 \in \ker T = \{0\} \Rightarrow \mathrm{v}_1 = \mathrm{v}_2$

$\therefore$ T is one-to-one. ■

VI. Participation and more (15pt) :   Name: ________________

<Fill this form, Print it, Bring it and submit it just before your Midterm Exam on AM 10:30, Oct. 20th)

1. (10pt) Participations

(1) QnA Participations Numbers <Check yourself> : each weekly (From Sat - next Friday)

Week 1 : 5   2: 5   3: 5   4: 5

Week 5 : 5   6: 5   7: 5   (8: 0)

Total# :   (Q: A:)

Online Participation : 31 / 33

Off-line Participation/ Absence : 12 / 13

(2) Your Special Contribution : including The number of your participations in Q&A with Finalized OK by SGLee (No.   ), Your valuable comments on errata (No.   ) or shared valuable informations and others (No.   )

(3) What are things that you have learned and recall well from the above participation?

2. (5pt) Project Proposal and/or Your Constructive suggestions

Title(Tentative), Goals and Objectives of your possible project:

** Linear Algebra in ??? Engneering ***

< Some of you made a good Project Proposal but not in general. Need to improve.>

SKKU LA 2015 PBL 보고서 발표 by 김** & 우**, http://youtu.be/hUDuQ8e8HsU

SKKU 선형대수학 PBL 보고서 발표 by 손** http://youtu.be/woyS_EYWiDs

SKKU 선형대수학 PBL 보고서 ppt 발표 by 박** http://youtu.be/E-5m65-8Ea8

Motivation and Significance of your possible project:

** My major and career ***

Working Plan:

** Team with ***

Web Resources (addresses) / References (book etc) : *****

선형대수학 자료실: http://matrix.skku.ac.kr/LinearAlgebra.htm

선형대수학 거꾸로 교실 자료 : http://matrix.skku.ac.kr/SKKU-LA-FL-Model/SKKU-LA-FL-Model.htm

* 선형대수학 강좌 운영방법 소개 동영상 : http://youtu.be/Mxp1e2Zzg-A

* 선형대수학 강좌 기록 일부 http://matrix.skku.ac.kr/2015-LA-FL/SKKU-LA-Model.pdf

http://matrix.skku.ac.kr/2015-LA-FL/Linear-Algebra-Flipped-Class-SKKU.htm

(Sample: http://www.prenhall.com/esm/app/ph-linear/kolman/html/proj_intro.html

http://home2.fvcc.edu/~dhicketh/LinearAlgebra/LinAlgStudentProjects.html

http://www.math.utah.edu/~gustafso/s2012/2270/projects.html

http://www2.stetson.edu/~mhale/linalg/projects.htm etc)

Etc: Write anything you like to tell me.

*****