Spring 2016, LA Final Exam - Solutions (1 hour In class Exam)

Course	Linear Algebra	GEDB003	41	Prof.	Sang-Gu Lee	Sign
Major		Year 학년		Student No. 학번		Name

※ Notice (Record of Lectures : http://matrix.skku.ac.kr/LA/)

1. Fill out the above boxes before you start this Exam. (학번, 이름 등을 기입하고 감독자 날인)
2. Honor Code: (시험 부정행위시 해당 교과목 성적이 "F" 처리됨은 물론 징계위원회에 회부될 수 있습니다.)
3. You can go out only after the permission from proctors.
   (감독위원의 지시가 있기 전에는 고사장 밖으로 나갈 수 없으며, 감독위원의 퇴실 지시가 있으면 답안지를 감독위원께 제출한 후에 퇴실하시기 바랍니다.)
4. You may use the following <Sage codes> in your answers. (Use only Math and English!!)

Total Score (150 pt)	
Offline Exam 135	Participation 15

```
var('a, b, c, d')                 # Define variables
eq1=3*a+3*b==12                   # Define equation1
eq2=5*a+3*b==13                   # Define equation2
solve([eq1, eq2], a,b)            # Solve eq's
A=matrix(QQ, 3, 3, [3, 0, 0, 0, 0, 2, 0, 3, 4]);      # Matrix
x=vector([3, 1, 2])               # Define vector x
A.augment(x)                      #  [A: x]
A.echelon_form()                  # Find RREF
A.inverse()                       # Find  inverse
A.det()                           # Find determinant
A.adjoint()                       # Find adjoint matrix
A.charpoly()                      # Find charct. ploy
A.eigenvalues()                   # Find eigenvalues
A.eigenvectors_right()            # Find eigenvectors
A.rank()                          # Find rank of A
A.right_nullity()                 # Find nullity of A
var('t')                          # Define variables
x=2+2*t                           # Define a parametric eq.
y=-3*t-2
bool( A== B)                      # Are A and B same?
```

```
var('x, y')                               # Define variables
f = 7*x^2 + 4*x*y + 4*y^2-23              # Define a function
implicit_plot( f, (x, -10, 10), (y, -10, 10))   # implicit Plot
parametric_plot((x,y), (t, -10, 10), rgbcolor='red')  # Plot
plot3d(y^2+1-x^3-x, (x, -pi, pi), (y, -pi, pi))   # 3D Plot
A=random_matrix(QQ,7,7)   # random matrix of size 7 over Q
F=random_matrix(RDF,7,7)  # random matrix of size 7 over R
P,L,U=A.LU()              # LU (P: Permutation M. / L, U
print P, L, U
y.norm()            # norm of y
h(x, y, z) = [x+2*y-z, y+z, x+y-2*z]
T = linear_transformation(U, U, h)   # L.T.
print T.kernel()                     # Find a basis for kernel(T)
C=column_matrix([x1, x2, x3])
D=column_matrix([y1, y2, y3])
aug=D.augment(C, subdivide=True)
Q=aug.rref()

[G,mu]=A.gram_schmidt()      # G-S
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B #
A.H   # conjugate transpose of A
A.jordan_form()   # Jordan Canonical Form of A
```

<Sample Sage Linear Algebra codes>

I. (1pt x 30= 30pt) True(T) or False(F).

1. (F) For any $n\times n$ matrix A with $n>1$, $\det(\operatorname{adj} A)=(\det A)^n$. (n-1)

2. (F) The Kernel of a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$ is a subspace of $\mathbb{R}^m$. ($\mathbb{R}^n$)

3. (F) The volume of parallelepiped which is generated by three vectors, $(8,0,3)$, $(0,-4,6)$, and $(4,2,-2)$ is 6. (16)

4. (F) If one replaces a matrix with its transpose, then the image, kernel, and nullity may change, and the rank is also changed.

5. (F) Let $T:\mathbb{R}^3\to\mathbb{R}^5$ be a linear transformation. Then T is one-to-one if and only if the rank is 5.

6. (F) If a linear transformation $T:\mathbb{R}^4\to\mathbb{R}^7$ is one-to-one, then the rank is 3 and the nullity is 4.

($\operatorname{rank}(A)=4$ and $\operatorname{nullity}(A)=7-\operatorname{rank}(A)=3$.)

7. (F) 1 is the dimension of a hyperplane in $\mathbb{R}^{11}$ perpendicular to a vector $\mathbf{a}$ in $\mathbb{R}^{11}$.

$$(\dim \mathbf{a}^{\perp}=\operatorname{nullity}(\mathbf{a}^T)=n-1=11-1=10,\ (\mathbf{a}\in\mathbb{R}^{11}))$$

8. (F) For given vectors $\mathbf{u}=(21,\ 11+36i,\ 0),\ \mathbf{v}=(14-7i,\ -101,\ 37i)$, Euclidean inner products $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{u}$ are same.

$$\mathbf{u}\cdot\mathbf{v}=\overline{(14-7i)}(21)+\overline{(-101)}(11+36i)+\overline{(37i)}(0)=-817-3489i$$

$$\mathbf{v}\cdot\mathbf{u}=\overline{(21)}(14-7i)+\overline{11+36i}(-101))+\overline{0}(37i)=-817+3489i=\overline{<\mathbf{u},\mathbf{v}>}$$

9. (F) The matrix $A=\begin{bmatrix}3+i & 2\\ 7 & 4-i\end{bmatrix}$ is Hermitian.

$$A^*=\begin{bmatrix}3-i & 7\\ 2 & 4+i\end{bmatrix}\neq\begin{bmatrix}3+i & 2\\ 7 & 4-i\end{bmatrix}=A\ :\ \text{not Hermitian}$$

10. (F) The matrix $A=\dfrac{1}{\sqrt{3}}\begin{bmatrix}-1 & 1-i\\ 1+i & 1\end{bmatrix}$ is not unitary.

$$AA^*=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}=I.\ \text{So } A \text{ is a unitary matrix.}$$

11. (F) $T: P_3 \to P_6$, $T(p(x)) = p(x)^2$ is a linear transformation

$T(p(x)+g(x)) = p(x)^2 + 2p(x)g(x) + g(x)^2 \neq p(x)^2 + g(x)^2 = T(p(x)) + T(g(x))$ $\therefore$ T is not a linear transformation. □

12. (T) Every nonzero vectors in the eigenspace of A corresponding to its eigenvalue λ is an eigenvector.

13. (T) For an $n \times n$ matrix A, A is invertible if and only if T_A is one-to-one and onto if and only if $\lambda = 0$ is not an eigenvalue of A.

14. (T) Let $\mathbf{a}$ be a nonzero column vector in $\mathbb{R}^n$. Then the standard matrix of $T(\mathbf{x}) = \text{proj}_{<\mathbf{a}>}\mathbf{x} = P\mathbf{x}$ is $P = \frac{1}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T$.

15. (T) If a square matrix A is symmetric, then eigenvectors of A corresponding to distinct eigenvalues are perpendicular to each other.

16. (T) The matrices $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}$ and A^TA have the same null space and row space.

17. (T) If an $m \times n$ matrix A has full column rank, then the pseudo-inverse of A is $A^\dagger = (A^TA)^{-1}A^T$.

18. (T) If A is a real symmetric matrix, then all the eigenvalues of A are real numbers.

19. (T) The range of the L.T. $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $T(A) = A + A^T$ is the set of all symmetric matrices in $M_2(\mathbb{R})$.

20. (T) For a nonzero vector $\mathbf{n} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$, the hyperplane Π: $\mathbf{n}^\perp = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{n} \cdot \mathbf{x} = 0,\ \mathbf{x} \in \mathbb{R}^n\}$ is called the orthogonal complement of $\mathbf{n}$. And the vector $\mathbf{n}$ is called the normal vector of the hyperplane Π.

21. (T) If there are n unknowns and k free variables in $A\mathbf{x} = \mathbf{b}$, then there are $n-k$ leading variables.

22. (T) Let $\mathbb{R}^n$, $\mathbb{R}^m$ are vector spaces and $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then $\ker T$ is a subspace of $\mathbb{R}^n$. Hence the subspace $\ker T$ is called a **kernel**.

23. (T) The following n vectors in $\mathbb{R}^n$, $\mathbf{x}_1 = (x_{11}, x_{12}, \ldots, x_{1n})$, ... , $\mathbf{x}_n = (x_{n1}, x_{n2}, \ldots, x_{nn})$ are linearly independent if and only if $\Delta = \begin{vmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ \vdots & & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{vmatrix} \neq 0$.

24. (T) If a linear transformation $T: V \to W$ is one-to-one and onto, then it is called an **isomorphism**. In this case, we say that V is isomorphic to W, denoted by $V \cong W$.

25. (T) Any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$.

26. (T) If the Jordan Canonical form of A is J_A, then

$$\text{rank}[(J_A - \lambda_i I)^k] = \text{rank}[(A - \lambda_i I)^k] \text{ and } \text{nullity}(A - \lambda_i I)^k = \text{nullity}(J_A - \lambda_i I)^k \text{ for all positive integer } k.$$

27. (T) Suppose the columns of $A_{m \times n} = [A^{(1)} | A^{(2)} | \cdots | A^{(n)}]_{m \times n}$ are orthonormal in $\mathbb{R}^m$. Then the column spaces of A^TA is an n dimensional subspace in $\mathbb{R}^n$.

28. (T) Let A be a 5×5 matrix with the only one eigenvalue λ with algebraic multiplicity of 5 and its dot diagram for λ is (· · · · ·). then a Jordan Canonical Forms of A is $J_A = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$.

29. (T) When we define the vector addition and the scalar multiplication on $M_2(\mathbb{R})$ as follows. M_2 forms a vector space.

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}, \quad k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}.$$

30. (T) The following subset is a subspace of $M_2(\mathbb{R})$ under the normal vector addition and the scalar multiplication?

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a + c = 0 \in \mathbb{R} \right\}.$$

II. (5pt x 4 = 20pt) State or Define (Choose/Mark 4 only: Fill the boxes and/or state).

1. [Cramer's Rule] For a system of linear equations, $A\mathrm{x} = \mathrm{b}$ where $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathrm{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$.

   If $|A| \neq 0$, the system of linear equations has a unique solution as follows:

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ \ldots,\ x_n = \frac{|A_n|}{|A|}$$

   where A_j $(j = 1, 2, \cdots, n)$ denotes the matrix A with the jth column replaced by the vector b. ■

2. [Rank-Nullity theorem] For any $A = [a_{ij}]_{m \times n}$, we have

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n.\ ■$$

3. [Isomorphism] A linear transformation $T: V \to W$ is called an isomorphism if T is one-to-one and onto. ■

4. [Normal matrix] An $n \times n$ real matrix $A = [a_{ij}] \in M_n(\mathbb{C})$ is called a normal matrix if

$$A^*A = AA^* \quad ■$$

.

5. [Jordan Canonical Form] Write how are you going to find a JCF of a given matrix.

Theorem 10.1.1

Let A be an $n \times n$ matrix with t $(1 \leq t \leq n)$ linearly independent eigenvectors. Then, A is similar to a matrix

$$J_A = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_t \end{bmatrix}_{n \times n}$$

where $U^*AU = J_A$ for some unitary matrix U. Furthermore, we have

$$J_k = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_{n_k \times n_k}, \quad (n_1 + n_2 + \cdots + n_t = n,\ 1 \leq k \leq t)$$

where each J_k, called a Jordan block, corresponds to an eigenvalue λ_i of A. The block diagonal matrix J_A is called the Jordan canonical form of A and each J_k are called Jordan blocks of J_A.

- The Jordan Canonical Form (JCF) of a matrix A is a block diagonal matrix composed of Jordan blocks, each with eigenvalues of A on its respective diagonal, 1's on its super-diagonal, and 0's elsewhere.

Theorem 10.1.2

The number of dots in the first r rows of the dot diagram for λ_i is equal to the dimension of solution space of $(A-\lambda_i I)^r \mathbf{x} = 0$ (i.e. the nullity of $(A-\lambda_i I)^r$).

- $\text{nullity}(A-\lambda_i I)^r = \text{nullity}(J_A-\lambda_i I)^r$.

Theorem 10.1.3

For $A \in M_n(\mathbb{C})$, let r_j denote the number of dots in the jth row of the dot diagram of λ_i. Then, the following are true.

(1) $r_1 = n - \text{rank}(A-\lambda_i I)$.

(2) If $j > 1$, $r_j = \text{rank}((A-\lambda_i I)^{j-1}) - \text{rank}((A-\lambda_i I)^j)$.

Proof By **Theorem 10.1.2**,

$$r_1 + r_2 + \cdots + r_j = \text{nullity}\ ((A-\lambda_i I)^j) = n - \text{rank}((A-\lambda_i I)^j)\ \text{(provided } j \geq 1)$$

Also, $r_1 = n - \text{rank}\,(A-\lambda_i I)$ and

$$\begin{aligned} r_j &= (r_1 + r_2 + \cdots + r_j) - (r_1 + r_2 + \cdots + r_{j-1}) \\ &= [n - \text{rank}((A-\lambda_i I)^j)] - [n - \text{rank}((A-\lambda_i I)^{j-1})] \\ &= \text{rank}((A-\lambda_i I)^{j-1}) - \text{rank}((A-\lambda_i I)^j),\ j > 1. \end{aligned}$$

(The number of dots in each row, r_j, means the number of blocks of size *at least* $j \times j$). ■

6. [Cayley Hamilton Theorem]

Every square matrix satisfies its own characteristic equation. ■

7. [Gershgorin's circle Theorem]

$A = [a_{ij}]$ is a square matrix of order n, $R_i = \sum_{j=1, i \neq j}^{n} |a_{ij}|$ is the sum of a absolute value of entries except for the diagonal entry of the ith row of A. Let $D(a_{ii}, R_i)$ be the closed disk centered at a_{ij} and with radius R_i. Such as disk is called by Gershgorin's disk(circle).

Every eigenvalue of A lies within at least one of Gershgorin's circle. ■

III. (10+5=15pt) Solve the following with Python/Sage Codes. (Answer or Fill the blanks)

1. (10 pts) You did define a matrix $A=\begin{bmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{bmatrix}$ and other works as following in a Sage Cell http://sage.skku.edu/.

And you found the following output. Please explain what you did in empty spaces as much as you can.

```
# Sage Cell http://sage.skku.edu/.
A=matrix(5,5,[29,14,2,6,-9,-47,-22,-1,-11,13,19,10,5,4,-8,-19,-10,-3,-2,8,7,4,3,1,-3])
#A = random_matrix(QQ, 7, 7)            1. What this means?  Make a 7 by 7 random matrix
y = matrix(QQ, 5, 1, [1, -1, 0, 1, 2])
F= A.augment(y)                         2. What this means? Make an augmented matrix [A : y]
print F
print F.echelon_form()                  3. What this means? Print an echelon form of F
print A.det()
print A.inverse()
print A.charpoly()
print A.eigenvalues()
print A.eigenvectors_right()
print A.jordan_form()                   4. What this means?  Print a JCF of A
var('x y')
f=2*x^2+sqrt(3)*x*y+y^2 -25
implicit_plot(f==0, (x,-10,10), (y,-10,10))     5. What this means?   Draw a graph of f(x, y)
```

[ANSWER: output]

```
[ 29  14   2   6  -9   1]
[-47 -22  -1 -11  13  -1]
[ 19  10   5   4  -8   0]
[-19 -10  -3  -2   8   1]
[  7   4   3   1  -3   2]

[  1   0   0   0   1   7]
[  0   2   0   0   2 -10]
[  0   0   1   0   3  10]
[  0   0   0   1   1   0]
[  0   0   0   0   8  19]

-16
```

6. What is this? Determinant of A

```
[ 49/4  23/4  -3/8  25/8  -5/2]
[-99/4 -47/4   1/8 -51/8     6]
[ 19/4   9/4  -1/8  11/8  -1/2]
[-19/4  -9/4   5/8  -7/8   1/2]
[ -5/4  -3/4  -5/8  -1/8   3/2]

x^5 - 7*x^4 + 16*x^3 - 8*x^2 - 16*x + 16
```

7. What is this? Characteristic Poly. of A

```
[-1, 2, 2, 2, 2]
```

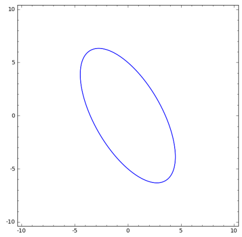
8. What is this? Eigenvalues of A are -1, and 2 (4 of them)

```
[(-1, [ (1, -2, 1/2, -1/2, 0) ], 1), (2, [ (1, 0, 3, -1, 3), (0, 1, 2, 0, 2) ], 4)]
```

9. What are the algebraic and geometric multiplicities of each eigenvalues?

-1 is an Eigenvalues of A and its corresponding eigenvector is(1, -2, 1/2, -1/2, 0) , **algebraic and geometric multiplicities of -1 is both 1.** 2 is an Eigenvalues of A and it has two linearly independent eigenvectors (1, 0, 3, -1, 3), (0, 1, 2, 0, 2), so **algebraic multiplicities of 2 is 4, but and geometric multiplicities of 2 is 2.**

```
[-1| 0  0  0| 0]
[--+--------+--]
[ 0| 2  1  0| 0]
[ 0| 0  2  1| 0]
[ 0| 0  0  2| 0]
[--+--------+--]
[ 0| 0  0  0| 2]
```



10. What this means? The graph of quadratic curve f is ellipse.

2. (5 pts) Find matrix P orthogonally diagonalizing matrix A and the diagonal matrix D such that $P^TAP = D$, using Sage.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

Solved by Sage(http://math3.skku.ac.kr/home/pub/415)

INPUT	A= matrix(QQ,3,3,[1,2,2,2,1,-2,2,-2,1]) print A.eigenvectors_left()
OUTPUT	[(-3, [(1, -1, -1)], 1), (3, [(1, 0, 1), (0, 1, -1)], 2)] : OK
INPUT (Gram-Schmidt Orthonormaliz-ation)	x1 = vector([1,-1,-1]) x2 = vector([1,0,1]) x3 = vector([0,1,-1]) z1 = 1/sqrt(3)*x1 y2 = x2 y3 = x3-((x3*y2)/(y2.norm())^2)*y2 z2 = 1/y2.norm()*y2 z3 = 1/y3.norm()*y3 P = column_matrix([z1,z2,z3]) print P print print P.transpose()*P
OUTPUT	[1/3*sqrt(3) 1/2*sqrt(2) 1/3*sqrt(3/2)] [-1/3*sqrt(3) 0 2/3*sqrt(3/2)] [-1/3*sqrt(3) 1/2*sqrt(2) -1/3*sqrt(3/2)] [1 0 0] [0 1 0] [0 0 1] : OK
INPUT	D = P.transpose()*A*P print D
OUTPUT	[-3 0 0] [0 3 0] [0 0 3] : OK

Ans)

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}, \text{ and } D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{ where } P^TAP = D \ . \quad \blacksquare$$

IV. (5pt x 10 = 50pts) Find or Explain (Fill the boxes) :

1. [Standard matrix of Projection] For $\mathbf{y}=(1,2,1)$ and $\mathbf{x}=(2,1,-1)$, find the standard matrix for $T(\mathbf{y})=\text{proj}_{<\mathbf{x}>}\mathbf{y}$.

Sol By theorem 7.5.2, the standard matrix P for projection can be derived as following.

$$P=\frac{1}{\mathbf{x}^T\mathbf{x}}\mathbf{x}\mathbf{x}^T=\frac{1}{[2\,1-1]\begin{bmatrix}2\\1\\-1\end{bmatrix}}\begin{bmatrix}2\\1\\-1\end{bmatrix}[2\,1-1]=\frac{1}{6}\begin{bmatrix}4&2&-2\\2&1&-1\\-2&-1&1\end{bmatrix}.\Rightarrow P\mathbf{y}=\frac{1}{6}\begin{bmatrix}4&2&-2\\2&1&-1\\-2&-1&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}.\quad\blacksquare$$

2. Let T_1 and T_2 be defined as follows $T_1(x_1,x_2,x_3)=(5x_1,\,-3x_1+2x_2,\,-x_1+4x_2)$ and $T_2(x_1,x_2,x_3)=(x_1+6x_2,\,-2x_3,\,x_1-x_3)$.

Then $T_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}5\\-3\\-1\end{bmatrix},\ T_1\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\2\\4\end{bmatrix},\ T_1\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\therefore[T_1]=\begin{bmatrix}5&0&0\\-3&2&0\\-1&4&0\end{bmatrix}$ and

$T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\1\end{bmatrix},\ T_2\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}6\\0\\0\end{bmatrix},\ T_2\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\-2\\-1\end{bmatrix}\quad\therefore[T_2]=\begin{bmatrix}1&6&0\\0&0&-2\\1&0&-1\end{bmatrix}$.

Find the standard matrix for each $T_2\circ T_1$. Checked by Sage: http://math3.skku.ac.kr/home/pub/334

Sol $[T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}5&0&0\\-3&2&0\\-1&4&0\end{bmatrix}\begin{bmatrix}1&6&0\\0&0&-1\\1&0&-1\end{bmatrix}=\begin{bmatrix}5&30&0\\-3&-18&-4\\-1&-6&-8\end{bmatrix}$ $\blacksquare$

3. Find the least square curve $y=a_0+a_1x+a_2x^2$ passing through the four points $(1,5)$, $(2,1)$, $(-1,-3)$, $(3,-2)$.

Sol From $y=a_0+a_1x+a_2x^2$, we have a linear system, $\begin{cases}5=a_0+a_1+a_2\\1=a_0+2a_1+4a_2\\-3=a_0-a_1+a_2\\-2=a_0+3a_1+9a_2\end{cases}$

Let $M=\begin{bmatrix}1&1&1\\1&2&4\\1&-1&1\\1&3&9\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$, and $\mathbf{b}=\begin{bmatrix}5\\1\\-3\\-2\end{bmatrix}$. $\Rightarrow M\mathbf{x}=\mathbf{b}\Rightarrow[M:\ \mathbf{b}]=\left[\begin{array}{ccc:c}1&1&1&5\\1&2&4&1\\1&-1&1&-3\\1&3&9&-2\end{array}\right]\Rightarrow\text{RREF}([M:\ \mathbf{b}])=\left[\begin{array}{ccc:c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]$

$\Rightarrow$ No solution, so we need the least square curve $y=a_0+a_1x+a_2x^2$.

Let $M\mathbf{x}'=\mathbf{b}'$ where $\mathbf{b}'=\text{proj}_W\mathbf{b}$ and W is a subspace of $\mathbb{R}^4$ spanned by the column vectors of M.

$\Leftrightarrow\ \mathbf{b}'\in\text{Col}(M)\Leftrightarrow\ \mathbf{b}-\mathbf{b}'=\mathbf{b}-M\mathbf{x}'\in\text{Col}(M)^{\perp}=\text{Null}(M^T)\ \Leftrightarrow\ M^T\cdot(\mathbf{b}-M\mathbf{x}')=0$

$\Leftrightarrow\ \mathbf{x}'=(M^TM)^{-1}M^T\mathbf{b}\quad\Rightarrow\ \mathbf{x}'=\begin{bmatrix}\frac{49}{22}\\\frac{747}{220}\\-\frac{73}{44}\end{bmatrix}\quad\therefore\ y=\frac{49}{22}+\frac{747}{220}x-\frac{73}{44}x^2$ $\blacksquare$

4. Find an invertible matrix P diagonalizing a given matrix A which has complex eigenvalues?

$$A=\begin{bmatrix}6&-4\\8&-2\end{bmatrix}$$

Sol $P^{-1}AP=C,\quad A-\lambda I=\begin{vmatrix}6-\lambda&-4\\8&2-\lambda\end{vmatrix}=\lambda^2-4\lambda-20=0\quad\therefore\ \lambda=2+4i,\ 2-4i$

1) For $\lambda_1=2+4i$, $\quad(A-\lambda_1I)\mathbf{x}_1=\begin{bmatrix}4-4i&-4\\8&-4-4i\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix},\quad b=(1-i)a\Rightarrow\mathbf{x}_1=\begin{bmatrix}1\\1-i\end{bmatrix}$.

2) For $\lambda_2=2-4i$, $\quad(A-\lambda_2I)\mathbf{x}_2=\begin{bmatrix}4+4i&-4\\8&-4+4i\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix},\quad b=(1+i)a\Rightarrow\mathbf{x}_2=\begin{bmatrix}1\\1+i\end{bmatrix}$.

$\therefore\ P=[\mathbf{x}_1:\mathbf{x}_2]=\begin{bmatrix}1&1\\1-i&1+i\end{bmatrix}$ and $P^{-1}AP=\begin{bmatrix}2+4i&0\\0&2-4i\end{bmatrix}$. $\blacksquare$

5. Compute $q(x)=\mathbf{x}^TA\mathbf{x}$ when $A=\begin{bmatrix}3&2&1\\-1&0&-4\\5&-2&1\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$

Sol $q(x,y,z)=\mathbf{x}^TA\mathbf{x}=[x\ y\ z]\begin{bmatrix}3&2&1\\-1&0&-4\\5&-2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=3x^2+z^2+xy+6xz-6yz.$ $\blacksquare$

6. Find a matrix P that eliminate the cross-product terms from the quadratic surface

$5x^2+6y^2+7z^2+4xy+4yz=1$ to $3(x')^2+6(y')^2+9(z')^2=1$ by properly rotating the axes.

Sol Let $q(x,y,z) = 5x^2+6y^2+7z^2+4xy+4yz = \mathbf{x}^TA\mathbf{x}$ where $A=\begin{bmatrix}5&2&0\\2&6&2\\0&2&7\end{bmatrix}$ and $\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}$.

$\Rightarrow \lambda_1=3, \lambda_2=6, \lambda_3=9 \Rightarrow \mathbf{v}_1=\frac{1}{3}\begin{bmatrix}2\\-2\\1\end{bmatrix},\ \mathbf{v}_2=\frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix},\ \mathbf{v}_3=\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$.

$\Rightarrow P=\frac{1}{3}\begin{bmatrix}2&2&1\\-2&1&2\\1&-2&2\end{bmatrix}$ and let $\mathbf{x}=P\mathbf{x}'$. Then we get $3\ (x')^2+6\ (y')^2+9\ (z')^2=1$ in a new axes $\mathbf{x}'=P^{-1}\mathbf{x}$. ■

7. The SVD of $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\sqrt3&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt6}&\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt3}&\frac{-1}{\sqrt3}&\frac{1}{\sqrt3}\end{bmatrix}$ is given. **Find its pseudo-inverse** $A^\dagger=V\Sigma'U^T$.

Sol Let $U=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}$, $\Sigma=\begin{bmatrix}\sqrt3&0&0\\0&1&0\end{bmatrix}$, $V^T=\begin{bmatrix}\frac{1}{\sqrt6}&\frac{2}{\sqrt6}&\frac{1}{\sqrt6}\\\frac{1}{\sqrt2}&0&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt3}&\frac{-1}{\sqrt3}&\frac{1}{\sqrt3}\end{bmatrix}$. Then $A=U\Sigma V^T$.

$\Rightarrow A^\dagger=(U\Sigma V^T)^\dagger = V\Sigma'U^T$ (where $\Sigma'=\begin{bmatrix}\Sigma_1^{-1}\\O\end{bmatrix}$ and $\Sigma_1^{-1}=\begin{bmatrix}\frac{1}{\sqrt3}&0\\0&1\end{bmatrix}$.)

$$=\begin{bmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\\\frac{2}{\sqrt6}&0&-\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt3}&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}\end{bmatrix}$$ ■

8. Find the conjugate transpose A^* of the matrix $A=\begin{bmatrix}4+2i&-5i\\3&10i\\-7i&6\\9&-7+4i\end{bmatrix}$

Sol $A^*=\overline{A}^T$. $\therefore A^*=\begin{bmatrix}\overline{4+2i}&\overline{3}&\overline{-7i}&\overline{9}\\\overline{-5i}&\overline{10i}&\overline{6}&\overline{-7+4i}\end{bmatrix}=\begin{bmatrix}4-2i&3&7i&9\\5i&-10i&6&-7-4i\end{bmatrix}$ ■

9. When we have an inner product on $\mathbb{R}^2$ as $<\mathbf{u},\mathbf{v}> = \mathbf{v}^TA\mathbf{u} = 6u_1v_1-2u_2v_1-2u_1v_2+3u_2v_2$.

Find A .

Sol $<\mathbf{u},\mathbf{v}>=\mathbf{v}^TA\mathbf{u}=\begin{bmatrix}v_1&v_2\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=au_1v_1+bv_1u_2+cv_2u_1+du_2v_2$.

$\therefore a=6,\ b=-2,\ c=-2,\ d=3,\ A=\begin{bmatrix}6&-2\\-2&3\end{bmatrix}$ □

10. Let $\mathbb{C}^3$ be the vector space with the Euclidean inner product. Transform $\mathbf{x}_1=(1,1,0)$, $\mathbf{x}_2=(0,1,2)$, $\mathbf{x}_3=(1,2,1)$. into an orthonormal basis by using the Gram-Schmidt process.

Sol Let $\mathbf{y}_1=\mathbf{x}_1$. Let W_1 be a subspace spanned by $\mathbf{u}_1$ and let

$\mathbf{y}_2=\mathbf{x}_2-\text{proj}_{W_1}\mathbf{x}_2=\mathbf{x}_2-\frac{\mathbf{x}_2\cdot\ \mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1=(0,1,2)-\frac{(0,1,2)\cdot\ (1,1,0)}{(\sqrt{|1|^2+|1|^2+|0|^2})^2}(1,1,0) = (0,1,2)-\frac{1}{2}(1,1,0)=\left(-\frac{1}{2},\frac{1}{2},2\right)$.

Let W_2 be a subspace spanned by $\mathbf{y}_1$ and $\mathbf{y}_2$ and let

$\mathbf{y}_3=\mathbf{x}_3-\text{proj}_{W_2}\mathbf{x}_3=\mathbf{x}_3-\frac{\mathbf{x}_3\cdot\ \mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1-\frac{\mathbf{x}_3\cdot\ \mathbf{y}_2}{\|\mathbf{y}_2\|^2}\mathbf{y}_2$

$=(1,2,1)-\frac{3}{2}(1,1,0)-\frac{5}{9}\left(-\frac{1}{2},\frac{1}{2},2\right)=\left(-\frac{2}{9},\frac{2}{9},-\frac{1}{9}\right)$

By normalizing $\mathbf{y}_1,\mathbf{y}_2$ and $\mathbf{y}_3$

$\mathbf{z}_1=\frac{\mathbf{y}_1}{\|\mathbf{y}_1\|}=\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2},0\right)$, $\mathbf{z}_2=\frac{\mathbf{y}_2}{\|\mathbf{y}_2\|}=\left(-\frac{\sqrt2}{6},\frac{\sqrt2}{6},\frac{2\sqrt2}{3}\right)$, $\mathbf{z}_3=\frac{\mathbf{y}_3}{\|\mathbf{y}_3\|}=\left(-\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right)$.

$\therefore$ This $Z=\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\}$ is an orthonormal basis. ■

V. (5pt x 4 = 20pts) Give a sketch of proof : <Choose and mark only 4 of 6!!>

1. Let $\mathbf{a}$ be a fixed vector and W be the set of all vectors orthogonal to $\mathbf{a}$, that is, $W=\{\mathbf{x}\in\mathbb{R}^3 \mid \mathbf{a}\cdot\mathbf{x}=0\}$. Show that W is a subspace of $\mathbb{R}^3$.

Proof [2 step Subspace test] Let $\mathbf{x}_1=\begin{bmatrix}d\\e\\f\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}g\\h\\i\end{bmatrix}$ in $W=\{\mathbf{x}\in\mathbb{R}^3 \mid \mathbf{a}\cdot\mathbf{x}=0\}$ and $k\in\mathbb{R}$. Then

1) $\mathbf{x}_1+\mathbf{x}_2=\begin{bmatrix}d+g\\e+h\\f+i\end{bmatrix}$ and $\mathbf{a}\cdot\mathbf{x}_1=0$, $\mathbf{a}\cdot\mathbf{x}_2=0 \Rightarrow \mathbf{a}(\mathbf{x}_1+\mathbf{x}_2)=\mathbf{a}\cdot\mathbf{x}_1+\mathbf{a}\cdot\mathbf{x}_2=0+0=0$ => $\mathbf{x}_1+\mathbf{x}_2\in W$

2) $k\mathbf{x}_1=\begin{bmatrix}kd\\ke\\kf\end{bmatrix}$, $\mathbf{a}(k\mathbf{x}_1)=k(\mathbf{a}\cdot\mathbf{x}_1)=k0=0$ => $k\,\mathbf{x}_1\in W$ $\therefore$ W is a subspace of $\mathbb{R}^3$. ■

2. [Properties of similar matrix] Suppose $B=P^{-1}AP$ for some P in $M_n(\mathbb{C})$, Then A and B has same eigenvalues with same algebraic multiplicities. [Hint: Show they have same characteristic polynomial.]

Proof $B=P^{-1}AP$ for some P in $M_n(\mathbb{C})$ => $\det(B-\lambda I_n)=\det(P^{-1}AP-\lambda I_n)$

=> $\det(P^{-1}AP-\lambda I_n)=\det(P^{-1}AP-\lambda P^{-1}P)=\det(P^{-1}(A-\lambda I_n)P)$

$=\det(P^{-1}(A-\lambda I_n)P)=\det(P^{-1})\det(A-\lambda I_n)\det(P)=\det(P^{-1})\det(P)\det(A-\lambda I_n)$

$=\det(I_n)\det(A-\lambda I_n)=\det(A-\lambda I_n)$

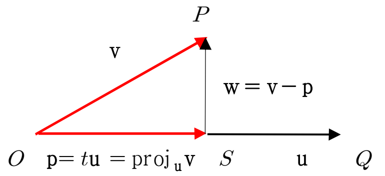
$\Rightarrow$ A, B has the same characteristic polynomial

$\Rightarrow$ A,B has same eigenvalues with same algebraic multiplicities. ■

3. [Theorem 9.2.1, (Cauchy-Schwarz inequality)] Let V be a complex inner product space.

For any $\mathbf{u},\mathbf{v}$ in V, $|\langle\mathbf{u},\mathbf{v}\rangle| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$.

Proof If $\mathbf{u}=0$, $|\langle\mathbf{u},\mathbf{v}\rangle|=0=\|\mathbf{u}\|\,\|\mathbf{v}\|$. Hence (1) holds. Let $\mathbf{u}\neq 0$ and $\mathbf{p}=\mathrm{proj}_{\langle\mathbf{u}\rangle}\mathbf{v}$, $\mathbf{w}=\mathbf{v}-\mathbf{p}$. Then $\langle\mathbf{w},\mathbf{p}\rangle=0$ and $\mathbf{p}=t\,\mathbf{u}=\dfrac{\langle\mathbf{v},\mathbf{u}\rangle}{\|\mathbf{u}\|^2}\mathbf{u}$. Thus we have the following.



$0 \le \langle\mathbf{w},\mathbf{w}\rangle=\langle\mathbf{w},\mathbf{v}-\mathbf{p}\rangle=\langle\mathbf{w},\mathbf{v}\rangle-\langle\mathbf{w},\mathbf{p}\rangle$

$=\langle\mathbf{w},\mathbf{v}\rangle=\langle\mathbf{v}-\mathbf{p},\mathbf{v}\rangle=\langle\mathbf{v},\mathbf{v}\rangle-\langle\mathbf{p},\mathbf{v}\rangle$

$=\|\mathbf{v}\|^2-t\langle\mathbf{u},\mathbf{v}\rangle=\|\mathbf{v}\|^2-\dfrac{\langle\mathbf{v},\mathbf{u}\rangle}{\|\mathbf{u}\|^2}\langle\mathbf{u},\mathbf{v}\rangle$.

Therefore $|\langle\mathbf{u},\mathbf{v}\rangle|^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$. ■

4. The vectors $p_1(t)=t^3+4t^2-3t+2$, $p_{2(t)}=2t^3+4t^2-t+1$, $p_3(t)=-t^3-2t+1$ in the vector space P_3 are linearly independent.

Sketch of Proof Let $c_1p_1(t)+c_2p_2(t)+c_3p_3(t)=0$,

=> $c_1=-1$, $c_2=1$, $c_3=1 \Rightarrow -p_1(t)+p_2(t)+p_3(t)=0$

So, the vectors are linearly dependent. ■

5. If $A\in M_n(\mathbb{C})$ is Hermitian. then for any vector $\mathbf{x}\in\mathbb{C}^n$, the product $\mathbf{x}^*A\mathbf{x}$ is a real number and every eigenvalue of A is a real number. ■

Sketch of Proof Let λ be an eigenvalue of A, that is, there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x}=\lambda\mathbf{x}$. By multiplying both sides by $\mathbf{x}^*$ on the left-hand side, we get $\mathbf{x}^*A\mathbf{x}=\mathbf{x}^*(\lambda\mathbf{x})=\lambda\mathbf{x}^*\mathbf{x}=\lambda\langle\mathbf{x},\mathbf{x}\rangle=\lambda\|\mathbf{x}\|^2$. Hence $\lambda=\dfrac{\mathbf{x}^*A\mathbf{x}}{\|\mathbf{x}\|^2}$. Since $\|\mathbf{x}\|^2$ is a nonzero real number, we just need to show that (1 by 1 scalar) $\mathbf{x}^*A\mathbf{x}$ is a real number. Note that $(\mathbf{x}^*A\mathbf{x})^*=\mathbf{x}^*A^*\mathbf{x}=\mathbf{x}^*A\mathbf{x}$.

Therefore, $\mathbf{x}^*A\mathbf{x}$ is a real number. ■

6. [Schur's Theorem]

A square matrix A is unitarily similar to an upper triangular matrix whose main diagonal entries are the eigenvalues of A. That is, there exists a unitary matrix U and an upper triangular matrix T such that

$$U^* A U = T = [t_{ij}] \in M_n(\mathbb{C}),\ t_{ij} = 0 (i > j),$$

where t_{ii}'s are eigenvalues of A.

Sketch of Proof

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. We prove this by mathematical induction. First, if $n=1$, then the statement holds because $A=[\lambda_1]$. We now assume that the statement is true for any square matrix of order less than or equal to $n-1$.

① Let $\mathbf{x}_1$ be an eigenvector corresponding to eigenvalue λ_1.

② By the Gram-Schmidt Orthonormalization, there exists an orthonormal basis for $\mathbb{C}^n$ including $\mathbf{x}_1$, say $S = \{\mathbf{x}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$.

③ Since S is orthonormal, the matrix $U_0 \equiv [\mathbf{x}_1 \ \vdots\ \mathbf{z}_2 \ \vdots\ \cdots \ \vdots\ \mathbf{z}_n]$ is a unitary matrix. In addition, since $A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$, the first column of AU_0 is $\lambda_1 \mathbf{x}_1$. Hence $U_0^*(AU_0)$ is of the following form:

$$U_0^* A U_0 = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline O & A_1 \end{array}\right]$$

where $A_1 \in M_{n-1}(\mathbb{C})$. Since $|\lambda I_n - A| = (\lambda - \lambda_1)\,|\lambda I_{n-1} - A_1|$, the eigenvalues of A_1 are $\lambda_2, \lambda_3, \ldots, \lambda_n$.

④ By the induction hypothesis, there exists a unitary matrix $\widehat{U}_1 \in M_{n-1}(\mathbb{C})$ such that

$$\widehat{U}_1^* A_1 \widehat{U}_1 = \begin{bmatrix} \lambda_2 & & * \\ 0 & & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

⑤ Letting $U_1 \equiv \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \widehat{U}_1 & \\ 0 & & & \end{array}\right] \in M_n(\mathbb{C})$, we get

$$(U_0 U_1)^* A (U_0 U_1) = U_1^* U_0^* A U_0 U_1 = \begin{bmatrix} \lambda_1 & & & * \\ & & & \\ 0 & \lambda_2 & & * \\ 0 & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Since $U \equiv U_0 U_1$ is a unitary matrix, the result follows. ■

VI. LA 2016-S, Participation and more (15pt) : Name: ________________

<Fill this form, Print it, Bring it and submit it just before your Final Exam on AM (9:00, April. 19th)

(시험전에 프린트하여, 빈칸을 채워서 제출하거나, 시험 중에 확인하여 채워서 시험 시간 중에 제출하면 됩니다.)

1. (10pt) Participations

(1) QnA Participations Numbers <Check yourself> : each weekly (From Sat - next Friday)

Week 1 :	2:	3:	4:
Week 5 :	6:	7;	8:
Week 9 :	10:	11:	12:
Week 13 :	14:	15:	

Total# : (Q: Anaser/Revision/Final:)

Online Participation : / 102 (1-15th week)

Off-line Participation/ Absence : /26 (Total 26 off line classes)

(2) Your Special Contribution :

: The number of your participations in Q&A with 'Finalized OK by SGLee' (No.),

(3) What are things that you have learned and recall well from QnA and PBL participation?

2. (5pt) Your Team Project

(1) Your Team Number ( ) and Team members name ( )

(2) What did you do on your team Project?

(3) What was your role in that process?

(4) What was you have learned from your Final PBL and Presentation?

3. (1pt, Bonus) Write anything you like to tell me.

* Have a great Summer break^^