Spring 2017, LA Final Comprehensive Exam (1 hour in class Exam) 채점용답안							Sign
Course	Linear Algebra	GEDB003		Prof.	Sang-Gu Lee		
Class # (mark) 분반	41 or 42	Year 학년		Student No. 학번		Name	

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4. You may use the following <Sage codes> in your answers. (Use only Math and English in your Final!!)

28 + 15 + 13 + 35 + 24 = 115 pts

Total Score	
Offline Exam 115	Participation 15

I. (2pt x 14= 28pt) True(T) or False(F).

1. (T) Let A be an $n\times n$ matrix. For any i, j $(1\le i, j\le n)$ $a_{1j}C_{j1}+a_{2j}C_{j2}+\cdots+a_{nj}C_{jn}=0$ where C_{ij} are cofactors.

2. (F) Let W be a 1 dimensional subspace of $\mathbb{R}^n$. Then $W=\mathbf{a}^{\perp}$ for some nonzero vector $\mathbf{a}\in\mathbb{R}^n$.

(Let W be a $n-1$ dimensional subspace of $\mathbb{R}^n$. Then $W=\mathbf{a}^{\perp}$ for some nonzero vector $\mathbf{a}\in\mathbb{R}^n$.)

3. (T) If $|A|\neq 0$, then $\text{rank}(AB)=\text{rank}(B)=\text{rank}(BA)$.

4. (F) Let $S=\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for V. For $r<n$, any subset $T=\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of V is linearly dependent.

(Let $S=\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for V. For $r>n$, any subset $T=\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of V is linearly dependent.)

5. (T) Let A be a square matrix of order n. Then A is diagonalizable if and only if the sum of the geometric multiplicities of eigenvalues of A is equal to n if and only if A have n linearly independent eigenvectors if and only if each eigenvalue λ of A has the same algebraic and geometric multiplicity.

6. (F) Suppose $S=\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$. If A is an singular matrix of order n, show that the set $\{A\mathbf{x}_1, \ldots, A\mathbf{x}_n\}$ is also a basis for $\mathbb{R}^n$.

(Suppose $S=\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis for $\mathbb{R}^n$. If A is an invertible matrix of order n, show that the set $\{A\mathbf{x}_1, \ldots, A\mathbf{x}_n\}$ is also a basis for $\mathbb{R}^n$.)

7. (T) Let the decomposition $A=U\Sigma V^T$ be the singular value decomposition (SVD) of an $m\times n$ $(m\ge n)$ matrix A where $\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_r$ are positive diagonal entries of $\Sigma=\begin{bmatrix}\Sigma_1 & O\\ O & O\end{bmatrix}$ and Σ_1 is nonsingular. Then $V^T(A^TA)V=\text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_r^2, 0, \ldots, 0)_{n\times n}$ and A can be expressed as $A=\sum_{j=1}^{r}\sigma_j\mathbf{u}_j\mathbf{v}_j^T$ where the columns of U, $\mathbf{u}_j$, are the left singular vectors of A and the columns of V, $\mathbf{v}_j$, are the right singular vectors of A. Also the $n\times m$ matrix $A^{\dagger}=V\Sigma' U^T$ is a pseudo-inverse of A, where $\Sigma'=\begin{bmatrix}\Sigma_1^{-1} & O\\ O & O\end{bmatrix}$. If A has full column rank, then the least squares solution to $A\mathbf{x}=\mathbf{b}$ is $\mathbf{x}=(A^TA)^{-1}A^T\mathbf{b}=A^{\dagger}\mathbf{b}$.

8. (F) Suppose $\mathbb{C}^n$ has the Euclidean inner product and U is a unitary matrix. Then $\|U\mathbf{x}\|=\|\mathbf{x}\|$ for $\mathbf{x}, \mathbf{y}\in\mathbb{C}^n$ and $\lambda=1$ for each and all eigenvalues λ of U.

(Suppose $\mathbb{C}^n$ has the Euclidean inner product and U is a unitary matrix. Then $\|U\mathbf{x}\|=\|\mathbf{x}\|$ for $\mathbf{x}, \mathbf{y}\in\mathbb{C}^n$ and $|\lambda|=1$ for each and all eigenvalues λ of U.)

9. (T) If $A\in M_n(\mathbb{C})$ is skew-Hermitian, then every eigenvalue of A is a pure imaginary number.

10. (F) The set of invertible matrices of order n is a subspace of the vector space M_n.

(The set of invertible matrices of order n is not a subspace of the vector space M_n.)

11. (T) If the columns of $A_{m\times n}=[A^{(1)}|A^{(2)}|\cdots|A^{(n)}]_{m\times n}$ are orthonormal in $\mathbb{C}^m$. Then the column spaces of $A^TA_{n\times n}$ is an n dimensional subspace in $\mathbb{C}^n$.

12. (F) If $f_1(x), f_2(x), \ldots, f_n(x)$ are $n-1$ times differentiable on the interval $(-\infty, \infty)$ and there exists $x_0\in(-\infty,\infty)$ such that Wronskian $W(x_0)=\begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0)\\ f_1'(x_0) & \cdots & f_n'(x_0)\\ \vdots & \vdots & \vdots\\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0)\end{vmatrix}$ is not zero, then these functions are linearly dependent. Conversely if

$W(x)=0$ for some x in $(-\infty,\infty)$, then f_1, ..., f_n are linearly independent.

(If $f_1(x), f_2(x), \ldots, f_n(x)$ are $n-1$ times differentiable on the interval $(-\infty,\infty)$ and there exists $x_0 \in (-\infty,\infty)$ such that Wronskian $W(x_0)=\begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{vmatrix}$ is not zero, then these functions are linearly independent.
Conversely if $W(x)=0$ for every x in $(-\infty,\infty)$, then f_1, ..., f_n are linearly dependent.)

13. (T) If A is an $n\times n$ symmetric and positive definite matrix, then $<\mathbf{u}, \mathbf{v}> = \mathbf{v}^T A \mathbf{u}$ defines an inner product on $\mathbb{R}^n$. The well known Euclidean inner product $\mathbf{u}\cdot \mathbf{v} = \mathbf{v}^T\mathbf{u} = \mathbf{v}^T I \mathbf{u}$ is its special case when $A=I_n$.

14. (F) Any n-dimensional real vector space is isomorphic to $\mathbb{C}^n$ and $P_n \cong \mathbb{R}^n$ and $M_{m\times n}(\mathbb{C}) \cong \mathbb{C}^{m\times n}$.

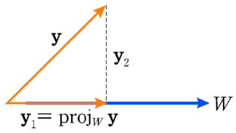
(Any n-dimensional complex vector space is isomorphic to $\mathbb{C}^n$ and $M_{m\times n}(\mathbb{C}) \cong \mathbb{C}^{m\times n}$.)

II. (3pt x 5 = 15pt) State or Define.

1. State more than 5 things that you know/can/find after you studied in our LA class.

...
Finding shortest distance between a point and plane (점과 평면사이의 최단거리 구하기),
Finding solutions of LSE using Gauss-Jordan Elimination and/or inverse matrix and/or Cramer's rule,
Finding a basis for a given vector space,
Finding a matrix representation of a given linear transformation,
Finding eigenvalues and eigenvectors,
Finding the singular value decomposition (SVD) of an $m\times n$ matrix A.
...

2. Let $S=\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for $\mathbb{R}^n$. Then we can obtain an orthonormal basis for $\mathbb{R}^n$ from S by **Gram-Schmidt orthonormalization process.** Fill the gap(box) in the process.



We first derive an orthogonal basis $T=\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ for $\mathbb{R}^n$ from the basis S as follows:

[Step 1] Take $\mathbf{y}_1 = \mathbf{x}_1$.

[Step 2] Let W_1 be a subspace spanned by $\mathbf{y}_1$ and let $\mathbf{y}_2 = \mathbf{x}_2 - \mathrm{proj}_{W_1}\mathbf{x}_2$ $= \mathbf{x}_2 - \dfrac{\mathbf{x}_2\cdot \mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1$.

[Step 3] Let W_2 be a subspace spanned by $\mathbf{y}_1$ and $\mathbf{y}_2$ and let

$$\mathbf{y}_3 = \mathbf{x}_3 - \mathrm{proj}_{W_2}\mathbf{x}_3 \qquad = \mathbf{x}_3 - \frac{\mathbf{x}_3\cdot \mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1 - \frac{\mathbf{x}_3\cdot \mathbf{y}_2}{\|\mathbf{y}_2\|^2}\mathbf{y}_2.$$

[Step 4] Repeat the same procedure to get

$$\mathbf{y}_k = \mathbf{x}_k - \mathrm{proj}_{W_{k-1}}\mathbf{x}_k \qquad = \mathbf{x}_k - \frac{\mathbf{x}_k\cdot \mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1 - \frac{\mathbf{x}_k\cdot \mathbf{y}_2}{\|\mathbf{y}_2\|^2}\mathbf{y}_2 - \cdots - \frac{\mathbf{x}_k\cdot \mathbf{y}_{k-1}}{\|\mathbf{y}_{k-1}\|^2}\mathbf{y}_{k-1} \text{ where } W_k = <\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k> \ (k=4, 5, \ldots, n).$$

=> $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is orthogonal. By taking $\mathbf{z}_k = \dfrac{\mathbf{y}_k}{\|\mathbf{y}_k\|}$ $(k=1, 2, \ldots, n)$, we have an orthonormal basis $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$ for $\mathbb{R}^n$.■

3. [Procedure for diagonalizing a diagonalizable matrix A]

- Step 1 : Find n linearly independent eigenvectors $\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \ldots, \mathbf{p}^{(n)}$ of A.
- Step 2 : Construct a matrix P whose columns are $\mathbf{p}^{(1)}, \mathbf{p}^{(2)}, \ldots, \mathbf{p}^{(n)}$ in this order.
- Step 3 : The matrix P diagonalize A and $P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

The matrix $A=\begin{bmatrix} 6 & -7 & -9 \\ 0 & -2 & -1 \\ 0 & 0 & -5 \end{bmatrix}$ has eigenvalues $\lambda_1=6, \lambda_2=-2, \lambda_3=5$ and corresponding eigenvectors $\mathbf{v}_1=(1,1,0), \mathbf{v}_2=(\frac{7}{8},1,0), \mathbf{v}_3=(\frac{34}{33},\frac{1}{3},1)$. Find matrix P diagonalizing A and the associated diagonal matrix D such that $D=P^{-1}AP$.

Solution $P^{-1}AP = D = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ where $P = [\mathbf{v}_1 : \mathbf{v}_2 : \mathbf{v}_3] = \begin{bmatrix} 1 & 7/8 & 34/33 \\ 1 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$. ■

4. [(General) Vector space]

If a set $V(\neq\phi)$ has **two well-defined binary operations,** vector addition (A) '+' and scalar multiplication (SM) '$\cdot$', and for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $h, k \in \mathbb{R}$, **two basic laws** A. $\mathbf{x}, \mathbf{y} \in V \Rightarrow \mathbf{x}+\mathbf{y} \in V$. SM. $\mathbf{x} \in V$, $k \in \mathbb{R} \Rightarrow k\mathbf{x} \in V$. and **the following eight laws** hold, then we say that the set V forms a vector space over $\mathbb{R}$ with the given two operations, and we denote it by $(V, +, \cdot)$ (simply V if there is no confusion). Elements of V are called **vectors**.

A1. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$. A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$.

A3. For any $\mathbf{x} \in V$, there exists a unique element $\mathbf{0}$ in V such that $\mathbf{x}+\mathbf{0} = \mathbf{x}$.

A4. For each element $\mathbf{x}$ of V, there exists a unique $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.

SM1. $k(\mathbf{x}+\mathbf{y}) = k\mathbf{x}+k\mathbf{y}$. SM2. $(h+k)\mathbf{x} = h\mathbf{x}+k\mathbf{x}$.

SM3. $(hk)\mathbf{x} = h(k\mathbf{x}) = k(h\mathbf{x})$. SM4. $1\mathbf{x} = \mathbf{x}$.

5. [Jordan Canonical Form] Write what you know about JCF and how are you going to find a JCF of a given matrix.

If a given matrix is diagonalizable, most computational problems involving that matrix and desired conclusions can be easily obtained. However, not every matrix is diagonalizable. In this case, we use a method for finding the Jordan Canonical Form of a non-diagonalizable matrix by a similarity transformation.

For *every* square matrix A (not necessarily diagonalizable), one can obtain a block-diagonal matrix called the **Jordan canonical form** matrix that is similar to A.

- Reference video: https://youtu.be/8fwPPOg8LW0 https://youtu.be/djd1XktKViA
- Practice site: http://matrix.skku.ac.kr/LA/Ch-10/ http://matrix.skku.ac.kr/JCF/

Theorem 10.1.1

Let A be an $n \times n$ matrix with t $(1 \le t \le n)$ linearly independent eigenvectors. Then, A is similar to a matrix

$$J_A = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_t \end{bmatrix}_{n\times n} \quad \text{(or}$$

where $U^* A U = J_A$ for some unitary matrix U. Furthermore, we have

$$J_k = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_{n_k \times n_k}, \quad (n_1+n_2+\cdots+n_t = n,\ 1 \le k \le t)$$

where each J_k, called a **Jordan block**, corresponds to an eigenvalue λ_i of A. The block diagonal matrix J_A is called the **Jordan canonical form** of A and each J_k are called Jordan blocks of J_A.

We can find JCF in Sage Cell http://sage.skku.edu/ (or http://math3.skku.ac.kr , http://math1.skku.ac.kr/ , https://cloud.sagemath.com etc). by using the following commends (or other ways ...) :

```
A = matrix(QQ, 5, 5, [7, -2, -2, 1, 0, 3, 0, -2, 1, 2, 12, -4, -3, 2, 0, 6, -8, -4, 6, 4, 1, -2, -2, 1, 6] )
A.jordan_form()    # Find Jordan Canonical Form of A
```

...

III. (8+5=13pts) Find, Compute or Explain (Fill the spaces) :

1. We did define a matrix $A=\begin{bmatrix} 7 & -2 & -2 & 1 & 0 \\ 3 & 0 & -2 & 1 & 2 \\ 12 & -4 & -3 & 2 & 0 \\ 6 & -8 & -4 & 6 & 4 \\ 1 & -2 & -2 & 1 & 6 \end{bmatrix}$ and other works as following in a Sage Cell http://sage.skku.edu/.

And you found the following output. Please explain what you did [in empty spaces] as much as you can.

```
# Sage Cell http://sage.skku.edu/ (or http://math3.skku.ac.kr , http://math1.skku.ac.kr/ , https://cloud.sagemath.com etc).
A = matrix(QQ, 5, 5, [7, -2, -2, 1, 0, 3, 0, -2, 1, 2, 12, -4, -3, 2, 0, 6, -8, -4, 6, 4, 1, -2, -2, 1, 6] )
#A = random_matrix(QQ, 100, 100)                1. What this means?
y = matrix(QQ, 5, 1, [1, -1, 0, 1, 2])
F= A.augment(y)
print F
print F.echelon_form()
print A.det()
print A.inverse()                          2. What this means?
print A.charpoly()
print A.eigenvalues()
print A.eigenvectors_right()               3. What this means?
print A.jordan_form()                      4. What this means?
var('x, y')
f=3*x^2-sqrt(3)*x*y-y^2 -25
implicit_plot(f==0, (x,-10,10), (y,-10,10))
[G,mu]=A.gram_schmidt()                    5. What this command means?
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B #
```

Evaluate (실행) **[ANSWER: output]**

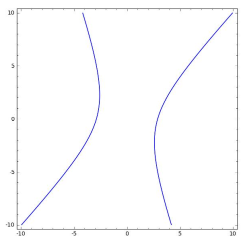
```
[[ 7 -2 -2  1  0  :  1]
[ 3  0 -2  1  2  : -1]
[12 -4 -3  2  0  : 0]
[ 6 -8 -4  6  4  : 1]
[ 1 -2 -2  1  6  : 2]  # augment matrix [ A : y ]
[    1     0     0     0     0   -5/4]
[    0     1     0     0     0 -29/12]
[    0     0     1     0     0   -9/2]
[    0     0     0     1     0 -49/12]
[    0     0     0     0     1 -13/12]
-144    #  det of A
[   -1   1/6   2/3  -1/12     0]
[ -4/3   2/3   2/3  -1/12  -1/6]
[   -4   1/3   7/3   -1/6     0]
[ -8/3   5/6   4/3   1/12  -1/3]
[ -7/6   1/6   2/3  -1/12   1/6]
x^5 - 16*x^4 + 95*x^3 - 260*x^2 + 324*x - 144
```

6. What the above is?

```
[6, 4, 3, 2, 1]    # eigenvalues of A
[(6, [(0, 1, 0, 2, 2)], 1), (4, [(1, 1, 2, 3, 1)], 1), (3, [(1, 1, 2, 2, 1)], 1), (2, [(0, 1, 0, 2, 0)], 1), (1, [(1, 1, 3, 2, 1)], 1)]
```

7. What are the algebraic and geometric multiplicities of eigenvalue 1 of A ?

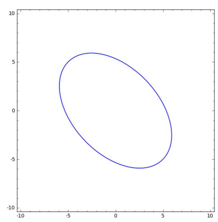
```
[ 6 | 0 | 0 | 0 | 0 ]
[-+-+-+-+-]
[ 0 | 4 | 0 | 0 | 0 ]
[-+-+-+-+-]
[ 0 | 0 | 3 | 0 | 0 ]
[-+-+-+-+-]
[ 0 | 0 | 0 | 2 | 0 ]
[-+-+-+-+-]
[ 0 | 0 | 0 | 0 | 1 ]
```



8. What the following vectors means?

```
[        29/1158*sqrt(1158)          7/579*sqrt(1158)          1/579*sqrt(1158)          1/193*sqrt(1158)         -3/386*sqrt(1158)]
[-301/2684*sqrt(1342/579)  127/1342*sqrt(1342/579) 1277/2684*sqrt(1342/579) -801/2684*sqrt(1342/579)   -75/244*sqrt(1342/579)]
[-859/9224*sqrt(2306/671) -235/4612*sqrt(2306/671) 2445/9224*sqrt(2306/671) 4219/9224*sqrt(2306/671) -143/9224*sqrt(2306/671)]
[      71/24*sqrt(10/1153)        -7/20*sqrt(10/1153)      611/120*sqrt(10/1153)     -251/120*sqrt(10/1153)      349/40*sqrt(10/1153)]
```

2. The graph of $7x^2+6xy+7y^2-200=0$ can be drown with Sage Command in http://sage.skku.edu.
var('x, y'), f=7*x^2+6*x*y+7*y^2-200, implicit_plot(f==0, (x,-10,10), (y,-10,10))



And $\mathrm{x}^T A\mathrm{x} - 200 = 0$ where $A=\begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}, \mathrm{x}=\begin{bmatrix} x \\ y \end{bmatrix}$ => $\lambda_1 = 10, \lambda_2 = 4$ => Unit eigenvectors

$$\mathrm{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathrm{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}, \ \mathrm{x} = P\mathrm{x}' \text{ and}$$

$$10x'^2 + 4y'^2 - 200 = 0 \quad \Rightarrow \quad \left(\frac{x'}{2\sqrt{5}}\right)^2 + \left(\frac{y'}{5\sqrt{2}}\right)^2 = 1.$$ (Answer the following questions))

(1) What is a new axis $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathrm{x}' = P^T\mathrm{x}$? (2) Explain this graph of $7x^2+6xy+7y^2-200=0$ as much as you can.

Answer (1) The new axis $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathrm{x}' = P^T \mathrm{x}$ is obtained by ratating 45 degree counterclockwise of the given $\begin{bmatrix} x \\ y \end{bmatrix}$ axis.

(2) The semiminor axis(단축) is $2\sqrt{5}$ and the semimajor axis(장축) is $5\sqrt{2}$ 인 타원을 그려서 시계방향으로 45도 회전시킨 그래프입니다. 즉 $(\frac{x'}{2\sqrt{5}})^2 + (\frac{y'}{5\sqrt{2}})^2 = 1$ in a new axis $\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathrm{x}' = P^T \mathrm{x}$ ■

IV. (5pt x 7 = 37pts) Find or Explain (Fill the boxes) :

1. Using the table below compute the dimension of $\mathrm{Row}(A)$, $\mathrm{Null}(A)$, $\mathrm{Col}(A)$, $\mathrm{Null}(A^T)$ for matrix A:

	(a)	(b)	(c)	(d)	(e)
Size of A	3×4	20×20	2×7	6×3	4×4
$\mathrm{rank}(A)$	3	13	2	3	4

Solution $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A) = \mathrm{rank}(A)$, $\dim \mathrm{Row}(A) + \dim \mathrm{Null}(A) = n$, $\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A^T) = m$.

	$\dim \mathrm{Row}(A)$	$\dim \mathrm{Null}(A)$	$\dim \mathrm{Col}(A)$	$\dim \mathrm{Null}(A^T)$
(a)	3	$4-3=1$	3	$3-3=0$
(b)	13	$20-13=7$	13	$20-13=7$
(c)	2	$7-2=5$	2	$2-2=0$
(d)	3	$3-3=0$	3	$6-3=3$
(e)	4	$4-4=0$	4	$4-4=0$

■

2. Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+z+w \\ x+y+2z-w \\ 2x+y+3z-2w \end{bmatrix} \text{ and } S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ x+3y+2z \\ 2x-y+3z \\ y-z \end{bmatrix} \ (\text{so } [S] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } [T] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & -2 \end{bmatrix}).$$

respectively. Find the matrix representation of the composition transformation $S \circ T$.

Sol $[S \circ T] = \begin{bmatrix} 3 & 1 & 4 & -1 \\ 8 & 5 & 13 & -6 \\ 7 & 2 & 9 & -3 \\ -1 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & -2 \end{bmatrix}$ ■

3. For $\mathrm{u}_1 = (1,2,3,5)$, $\mathrm{u}_2 = (2,3,5,1)$, $\mathrm{u}_3 = (1,3,6,2)$, $\mathrm{u}_4 = (1,3,2,1)$, $\mathrm{v}_1 = (2,1,3,4)$, $\mathrm{v}_2 = (4,3,2,1)$, $\mathrm{v}_3 = (5,2,1,3)$, $\mathrm{v}_4 = (3,3,2,1)$, let $\alpha = (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)$, $\beta = (\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4)$, which are ordered bases for $\mathbb{R}^4$.

(1) Find the transition matrix $[I]_\beta^\alpha$.

(2) Find $[\mathrm{w}]_\alpha = P[\mathrm{w}]_\beta$ using the transition matrix $[I]_\alpha^\beta$ when $[\mathrm{w}]_\beta = (3,2,1,3)$.

Solution

(1) Let $A = \begin{bmatrix} 1\,2\,1\,1 & \vdots & 2 & \vdots & 4 & \vdots & 5 & \vdots & 3 \\ 2\,3\,3\,3 & \vdots & 1 & \vdots & 3 & \vdots & 2 & \vdots & 3 \\ 3\,5\,6\,2 & \vdots & 3 & \vdots & 2 & \vdots & 1 & \vdots & 2 \\ 5\,1\,2\,1 & \vdots & 4 & \vdots & 1 & \vdots & 3 & \vdots & 1 \end{bmatrix}$. $P = [I]_\beta^\alpha = [[\mathrm{v}_1]_\alpha : [\mathrm{v}_2]_\alpha : [\mathrm{v}_3]_\alpha : [\mathrm{v}_4]_\alpha]$ comes from RREF of $A \Rightarrow$

$$B = \begin{bmatrix} 1\,0\,0\,0 & \vdots & 26/25 & \vdots & 27/50 & \vdots & 34/25 & \vdots & 9/25 \\ 0\,1\,0\,0 & \vdots & 33/25 & \vdots & 141/50 & \vdots & 97/25 & \vdots & 47/25 \\ 0\,0\,1\,0 & \vdots & -21/25 & \vdots & -117/50 & \vdots & -89/25 & \vdots & -39/25 \\ 0\,0\,0\,1 & \vdots & -21/25 & \vdots & 4/25 & \vdots & -14/25 & \vdots & 11/25 \end{bmatrix} \quad \therefore P = [I]_\beta^\alpha = \begin{bmatrix} 26/25 & 27/50 & 34/25 & 9/25 \\ 33/25 & 141/50 & 97/25 & 47/25 \\ -21/25 & -117/50 & -89/25 & -39/25 \\ -21/25 & 4/25 & -14/25 & 11/25 \end{bmatrix}$$ ■

(2) $[\mathrm{w}]_\alpha = P[\mathrm{w}]_\beta = \begin{bmatrix} 26/25 & 27/50 & 34/25 & 9/25 \\ 33/25 & 141/50 & 97/25 & 47/25 \\ -21/25 & -117/50 & -89/25 & -39/25 \\ -21/25 & 4/25 & -14/25 & 11/25 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 166/25 \\ 478/25 \\ -386/25 \\ -36/25 \end{bmatrix}$ ■

4. Find the SVD(Singular Value Decomposition) of $A=\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}$ where $AA^T=\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}=\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}$ and $A^TA=\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}$.

Sol The singular values of A are $\sigma_1=\sqrt{\lambda_1}=\sqrt{3}, \sigma_2=\sqrt{\lambda_2}=1$ and $\Sigma=\begin{bmatrix}\sqrt{3}&0\\0&1\\0&0\end{bmatrix}$.

Unit eigenvectors of A^TA corresponding to eigenvalues $\lambda_1=3, \lambda_2=1$ are $v_1=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$, $v_2=\begin{bmatrix}\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$, respectively.

Unit eigenvectors of AA^T corresponding to eigenvalues $\lambda_1=3, \lambda_2=1, \lambda_3=0$ are $u_1=\begin{bmatrix}\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{bmatrix}$, $u_2=\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{-1}{\sqrt{2}}\end{bmatrix}$, $u_3=\begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{-1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$, resp..

=> $U=\begin{bmatrix}\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{2}{\sqrt{6}}&0&\frac{-1}{\sqrt{3}}\\\frac{1}{\sqrt{6}}&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\end{bmatrix}$, $V=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$ => $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{2}{\sqrt{6}}&0&\frac{-1}{\sqrt{3}}\\\frac{1}{\sqrt{6}}&\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\sqrt{3}&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$ ■

5. The SVD of $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{3}&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}}&\frac{2}{\sqrt{6}}&\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&0&\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\end{bmatrix}$ is given. Find its pseudo-inverse $A^{\dagger}=V\Sigma' U^T$.

Sol Then $A=U\Sigma V^T. \Rightarrow A^{\dagger}=(U\Sigma V^T)^{\dagger}=V\Sigma' U^T$ (where $\Sigma'=\begin{bmatrix}\Sigma_1^{-1}\\O\end{bmatrix}$ and $\Sigma_1^{-1}=\begin{bmatrix}\frac{1}{\sqrt{3}}&0\\0&1\end{bmatrix}$.)

$$=\begin{bmatrix}\frac{1}{\sqrt{6}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\\\frac{2}{\sqrt{6}}&0&-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{6}}&-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{3}}&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}-\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}\end{bmatrix}$$ ■

6. Find the least square line $y=a_0+a_1x$ passing through the four points (1,2), (2,3), (3,4), (4,5).

Sol From $y=a_0+a_1x$, we have a linear system $\begin{cases}2=a_0+a_1\\3=a_0+2a_1\\4=a_0+3a_1\\5=a_0+4a_1\end{cases}$. Let $M=\begin{bmatrix}1&1\\1&2\\1&3\\1&4\end{bmatrix}$, $x=\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$, and $b=\begin{bmatrix}2\\3\\4\\5\end{bmatrix}$ ⇒ $Mx=b$.

$$\Rightarrow M^TMx=M^Tb \Leftrightarrow x'=(M^TM)^{-1}M^T b \Rightarrow x'=\begin{bmatrix}\frac{3}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{5}\end{bmatrix}\begin{bmatrix}1&1&1&1\\1&2&3&4\end{bmatrix}\begin{bmatrix}2\\3\\4\\5\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \quad \therefore \; y=1+x$$ ■

7. Let $\mathcal{E}([a,b],\mathbb{C})$ be the set of all continuous functions from the interval $[a,b]$ to the complex set $\mathbb{C}$. If the addition and scalar multiple of these functions are defined as $(f+g)(x)=f(x)+g(x)$, $(cf)(x)=cf(x)$, $c\in C$ for $f(x), g(x)\in\mathcal{E}([a,b],\mathbb{C})$, then $\mathcal{E}([a,b],\mathbb{C})$ is **a complex vector space with respect to these operations.** Then $f(x)=f_1(x)+if_2(x)$ in $\mathcal{E}([a,b],\mathbb{C})$ where $f_1(x)$, $f_2(x)$ are continuous functions from $[a,b]$ to $\mathbb{R}$.

For $f(x)$, $g(x)\in\mathcal{E}([a,b],\mathbb{C})$, define the following inner product $<f(x), g(x)>=\int_a^b \overline{g(x)}f(x)\,dx$. Then $\mathcal{E}([a,b],\mathbb{C})$ **is a complex inner product space.** Let $u=f(x)$, $v=g(x)\in\mathcal{E}([a,b],\mathbb{C})$. With the above inner product, **the Cauchy-Schwarz inequality** is given by

$$\left|\int_a^b \overline{g(x)}f(x)\,dx\right| = |<u,v>| \le \|u\|\,\|v\| = \left(\int_a^b |f(x)|^2dx\right)^{\frac{1}{2}}\left(\int_a^b |g(x)|^2dx\right)^{\frac{1}{2}}.$$

V. (6pt x 4 = 24pts) Give a sketch of proof or Explain or Fill the Box :

1. A linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\text{rank}(A) = \text{rank}\,[A \,\vdots\, \mathbf{b}]$.

Proof Let $A = [a_{ij}]_{m\times n}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$. Then the linear system $A\mathbf{x} = \mathbf{b}$ can be written as

$$x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}\end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn}\end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m\end{bmatrix}. \qquad (1)$$

Hence we have the following:

$A\mathbf{x} = \mathbf{b}$ has a solution.

⇔ There exist $x_1, x_2, \ldots, x_n$ satisfying the linear system (1).

⇔ $\mathbf{b}$ is a linear combination of the columns of A.

⇔ $\mathbf{b} \in \text{Col}(A)$

⇔ $\text{rank}(A) = \text{rank}\,[A \,\vdots\, \mathbf{b}]$. ■

2. For matrices A, B with multiplication AB defined, show $\text{Null}(B) \subseteq \text{Null}(AB)$.

Proof $\forall\, \mathbf{x} \in \text{Null}(B) \Rightarrow B\mathbf{x} = \mathbf{0}$ [Show $(AB)\mathbf{x} = \mathbf{0}$]

$\Rightarrow (AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$. $\therefore\ \mathbf{x} \in \text{Null}(AB)$. This implies $\text{Null}(B) \subseteq \text{Null}(AB)$. ■

3. Let A be a square matrix. Then A is orthogonally diagonalizable if and only if the matrix A is symmetric.

Proof (⇒) Suppose A is orthogonally diagonalizable. Then there exists an orthogonal matrix P and a diagonal matrix D such that $P^TAP = D$. [Show $A = A^T$] Since $D = D^T$, we have $P^TAP = D = D^T = (P^TAP)^T = P^TA^TP$.

Hence $P^TAP = P^TA^TP$ ⇔ $P(P^TAP)P^T = P(P^TA^TP)P^T$

⇔ $(PP^T)A(PP^T) = (PP^T)A^T(PP^T)$

⇔ $A = A^T$. Therefore, A is symmetric.

(⇐) http://www.maths.manchester.ac.uk/~peter/MATH10212/notes10.pdf ■

4. If $A = \overline{A}^T = A^*$ (Hermitian), then all the eigenvalues of A are real numbers.

Proof Let λ be an eigenvalue of A, that is, there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. By multiplying both sides by $\mathbf{x}^* = \overline{\mathbf{x}}^T$ on the left-hand side, we get $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*(\lambda\mathbf{x}) = \lambda\mathbf{x}^*\mathbf{x} = \lambda < \mathbf{x}, \mathbf{x} > = \lambda\,||\mathbf{x}||^2$. Hence $\lambda = \dfrac{\mathbf{x}^*A\mathbf{x}}{||\mathbf{x}||^2}$. Since $||\mathbf{x}||^2$ is a nonzero real number, we just need to show that $\mathbf{x}^*A\mathbf{x}$ is a real number. [Show $\overline{\mathbf{x}^*A\mathbf{x}} = \mathbf{x}^*A\mathbf{x}$]

$$\overline{\mathbf{x}^*A\mathbf{x}} = \overline{\mathbf{x}^*}\,\overline{A\mathbf{x}} = [\mathbf{x}^T(\overline{A\mathbf{x}})]_{1\times 1} = [\mathbf{x}^T(\overline{A\mathbf{x}})]^T_{1\times 1} \quad (\because\ \mathbf{x}^T(\overline{A\mathbf{x}}) \text{ is a scalar})$$

$$= (\overline{A\mathbf{x}})^T\mathbf{x} = (\overline{A}\,\overline{\mathbf{x}})^T\mathbf{x} = \overline{\mathbf{x}}^T\,\overline{A}^T\mathbf{x} \qquad = \mathbf{x}^*A\mathbf{x}. \quad (\because\ A = \overline{A}^T = A^*)$$

Therefore, $\mathbf{x}^*A\mathbf{x}$ and λ are all real numbers. ■

Class #	Only for Class 42

VI. LA 2017-S, Participation and more (15pt) :

Name: ____________________

<Do Fill this form, Print it, Bring it and submit it just before your Midterm Exam on AM (9:00, June 20th) at 32255

(시험전에 프린트하여, 빈칸을 채워서 제출하거나, 시험 중에 확인하여 채워서 시험 시간 중에 제출하면 됩니다.)

1. (8 pt) Participations

(1) QnA Participations Numbers <Check yourself> : each week (From Sat - next Friday)

Week 1 :	2:	3:	4:
Week 5 :	6:	7;	8:
Week 9 :	10:	11:	12:
Week 5 :	13:	14:	15:

Total# : 45+ (Q: A(답변/수정:):)

Online Attendence/Participation : 95% / 100 % (1-15th week)

Off-line Attendence : 95% /100% (Total 26 off line classes), Absence number :

2. (2 pt) Yes I did fill out OUR LA class survey in http://bit.ly/** (Yes, No)

3. (5pt) Your QnA (or Personal/Team Project)

(1) Write (sketch) a couple of Your Special Contributions (via QnA/PBL/Project participation).

(2) What was your main role in that learning process (Leader or Individual)?

(3) What was you have learned from your Final PBL(QnA, Discussion/Project, Presentation)?

3. (1pt, Bonus) Write anything you like to tell me/others.

* Have a great Summer break^^

- 연습지 이므로 띄어서 계산에 사용하시면 됩니다 -

[Reference] Use the following Table to fill out or write the meaning of your SageMath Work

var('a, b, c, d') # Define variables
 	
eq1=3*a+3*b==12 # Define equation1
eq2=5*a+2*b==13 # Define equation2
solve([eq1, eq2], a,b) # Solve eq's
 	
A=matrix(QQ, 3, 3, [3, 0, 0, 0, 0, 2, 0, 3, 4]); # define Matrix
x=vector([3, 1, 2]) # Define vector x
A.augment(x) # [A: x]
A.echelon_form() # Find RREF
A.inverse() # Find inverse
A.det() # Find determinant
 	
A.adjoint() # Find adjoint matrix
 	
A.charpoly() # Find charct. ploy
A.eigenvalues() # Find eigenvalues
A.eigenvectors_right() # Find eigenvectors
 	
A.rank() # Find rank of A
A.right_nullity() # Find nullity of A
 	
var('t') # Define variables
x=2+2*t # Define a parametric eq.
y=-3*t-2
bool(A== B) # Are A and B same?
	var('x, y') # Define variables
	f = 7*x^2 + 4*x*y + 4*y^2-23 # Define a function
	implicit_plot(f, (x, -10, 10), (y, -10, 10)) # implicit Plot
	parametric_plot((x,y), (t, -10, 10), rgbcolor='red') # Plot
	
	plot3d(y^2+1-x^3-x, (x, -pi, pi), (y, -pi, pi)) # 3D Plot
	
	A=random_matrix(QQ,7,7) # random matrix of size 7 over Q
	
	F=random_matrix(RDF,7,7) # random matrix of size 7 **over R**
	
	P,L,U=A.LU() # LU (P: Permutation M. / L, U
	print P, L, U
	
	h(x, y, z) = [x+2*y-z, y+z, x+y-2*z]
	T = linear_transformation(U, U, h) # L.T.
	print T.kernel() # Find a basis for kernel(T)
	C=column_matrix([x1, x2, x3])
	D=column_matrix([y1, y2, y3])
	aug=D.augment(C, subdivide=True)
	Q=aug.rref()
	
	[G,mu]=A.gram_schmidt() # G-S on Process, find G
	B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B
	# Find a matrix B with nomalized vectors in it and show B
	
	A.H # conjugate transpose of A
	
	A.jordan_form() # Find Jordan Canonical Form of A
	
	<Sample Sage Linear Algebra codes>