Fall 2013, Calculus II, Midterm Exam (Solution)					Sign
Course	**Calculus II** with Sage	GEDB021-42	Prof.	Sang-Gu Lee	

Major		Year(학년)		S.N(학번)		Name	

※ Notice (수험생 유의사항)

1. Write your name and get the professor's signature. 답안 작성전에 이름 등을 빠짐없이 기입하고 감독자 날인을 받으세요.
2. Keep your Honor code. If not, a serious penalty will be given.

I (2*18=36pts)	II (3*6=18)	III (2*19=38)	VI (3+5=8)	

Score (150)	
참여 및 PBL	EXAM
/50	/100

http://matrix.skku.ac.kr/Cal-Book/ http://matrix.skku.ac.kr/Mobile-Sage-G/sage-grapher.html http://matrix.skku.ac.kr/cal-lab/Math-CAS.htm

```
var('t')                                  # 변수정의 (매개변수방정식 )
x=2+2*t; y=-3*t-2
parametric_plot((x, y), (t, -10, 10), rgbcolor='red')  # 직선 Plot

var('a, b, c, d, x, y, z, r, n') # 변수정의 (매개변수방정식)
f(x)=exp(-2*x)                            # 함수 정의
g(x)=f.taylor(x,2,4)                      # 테일러전개

plot(f(x), (x, -2, 2))                    # 함수의 그래프
implicit_plot(sin(x)-2*y==3, (x, -2, 2), (y, -2, 2)) # 음함수 그래프

solve(f(x)==0, x)                 # Solve 방정식 풀이
diff(f(x), x, 2)                  # f(x)의 2계 도함수
integral(f(x), x)                 # x에 관한 부정적분
integral(f(x), x, -2, 3)          # [-2, 3]까지의 정적분
limit(f(x), x=2)                  # 극한
limit(f(x), x=2, dir='+')         # 우극한
limit(f(x), x=2, dir='-')         # 좌극한
limit(f(x), x=+oo)                # +infinity 에서의 극한
limit(f(x), x=-oo)                # -infinity 에서의 극한
limit(ln(3*n)/ln(5*n), n=+oo)

a(n) = ((e)/10)^n
sum(a(n), n, 1, +oo)  # 급수의 계산

u(n)=1/(n*2^n)
rho=limit(abs(u(n+1)/u(n)), n=+oo)
R=1/rho; R                        # 수렴반경

x = (3, -4, 5)
a=sqrt(x[0]^2)              #  yz평면과의 거리
b=sqrt(x[1]^2)              #  zx평면과의 거리
c=sqrt(x[2]^2)              #  xz평면과의 거리
d=sqrt(x[1]^2 +x[2]^2)      #  x 축과의 거리
e=sqrt(x[0]^2 +x[2]^2)      #  y 축과의 거리
f=sqrt(x[0]^2 +x[1]^2)      #  z 축과의 거리

f.partial_fractions(x)            # 부분분수
find_root(f(x), a, b)             # [a, b] 사이에서 근사해 구하기
```

```
a=vector(QQ, [2, -4, 3]);d= -2
a.norm()                                  # 벡터의 크기
distance=abs(a.dot_product(a)+d)/a.norm()    # 거리

r(t)=(t-t^3, t^2, 0)          # 벡터함수 정의
dr=diff(r(t), t)
ddr=diff(r(t), t, 2)
T=dr.dot_product(ddr)/dr.norm()       #.dot_product()는 내적
N=(dr.cross_product(ddr)).norm() / dr.norm()   #.cross_product()는 외적

var('t')                      # 변수정의 (매개변수방정식)
r(t)=(t-t^3, t^2, 0)          # 벡터함수 정의
dr=diff(r(t),t)
g(t)=sqrt((dr[0]^2+dr[1]^2+dr[2]^2).simplify_trig());
k=integral(g(t),t,-5,5)     # arc length
S=integral(r(t), t)           # integral은 적분하는 명령어
v=diff(r(t), t)             # diff는 t에 관해 도함수 구하는 명령어
a=diff(v, t)                # diff(r(t), 2) 와 a는 가속도 벡터함수
s= v.norm()      # s는 속도의 크기, .norm()은 벡터의 크기를 구한다.
v2=v.subs(t=2)         # v2는 t=2 일 때 속도
a2=a.subs(t=2)       # a2는 t=2 일 때 가속도
p1=parametric_plot(r(t), (t, 0, 3)) # parametric_plot은 매개변수함수 그림
p2=line([r(2), r(2)+v2], color='red') # line은 선분을 두 점을 잇는 선분
p3=line([r(2), r(2)+a2], color='green')
show(p1+p2+p3)

var('x, y, z')
p1=implicit_plot3d(x^2+y^2==5, (x, -3, 3), (y, -3, 3),  (z, -3, 3))
p2=implicit_plot3d(x*y==z,  (x,  -3,  3),  (y,  -3,  3),  (z,  -3,  3), rgbcolor='green', opacity=0.4)
show(p1+p2)

contour_plot(f(x, y), (x, -1, 1), (y, -1, 1), fill=False, cmap='hsv', axes=True, labels=True)  # level curve 그리기
```

I. (2pt x 18 = 36) Mark True(T) or False(F) in the blanks ( ).

1. (   ) ***Archimedean Property*** : Given any real number x, there exists an integer N_x such that $N_x < x$.

2. (   ) ***Squeeze (Sandwich) Property:*** If $a_n \le b_n \le c_n$ is true for all $n \ge 55$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

3. (F) ***Monotonic Sequence Theorem:*** Every bounded, monotonic sequence is divergent.

4. (F) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

5. (　　) The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

6. (　　) Let Σa_n and Σb_n be series of positive terms. Assume that $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ exists.

Then Σa_n converges if and only if Σb_n converges. Σa_n diverges if and only if Σb_n diverges.

7. (F) A series Σa_n is absolutely convergent if Σa_n converges, but $\Sigma |a_n|$ diverges.

8. (　　) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn on the convergence or divergence of Σa_n.

9. (　　) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive; that is, no conclusion can be drawn on the convergence or divergence of Σa_n.

10. (　　) The polynomial $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the n**th degree Taylor polynomial of** f **at** a when $R_n(x)$ is the **remainder** of the Taylor series of f about a. And f is equal to the sum of its Taylor series on the interval $|x-a| < R$ if and only if $\lim_{n\to\infty} R_n(x) = 0$, for $|x-a| < R$.

11. (F) We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, for all x, and $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, for all x in $(-1, 1)$.

12. (F) The cross product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$.

13. (F) The distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = (a, b, c)$ (See Figure). Thus, $D = |\text{proj}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

And the vector $\text{orth}_{\mathbf{n}} \mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{n}} \mathbf{b}$ known as orthogonal projection of $\mathbf{b}$, is orthogonal to $\mathbf{n}$.

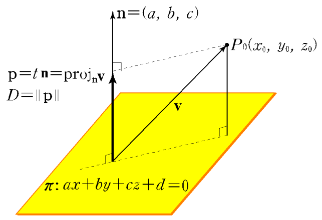

Figure $\text{proj}_{\mathbf{n}} \mathbf{b} = t\mathbf{n}$

14. (　　) If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

15. (　　) If $\mathbf{r}(t)$ is a vector function and $|\mathbf{r}(t)| = c$, where c is a constant for all t, then $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$ for all t.

16. (　　) Let C be a smooth curve defined by the vector function $\mathbf{r}$, and the unit tangent vector $\mathbf{T}(t)$ is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$. The **curvature** κ of a curve $\mathbf{r}$ is $\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$ and we can compute the curvature as $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ since $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$. If C is smooth curve defined by the vector function $\mathbf{r}$, then the curvature κ can be expressed as

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

17. (　　) The principal unit normal vector $\mathbf{N}(t)$ (or simply unit normal) for a smooth curve in space is $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.

18. (　　) Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. Then the graph of f is a set of points graph $f = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D \text{ and } z = f(x, y)\}$. One way to plot is to look at set of points $(x, y, f(x, y) = c)$ for various values of c, which is the intersection of the surface and the plane $z = c$. The intersection is called the **level curve** at the level $z = c$. If we know these level curves for all values of c, then we can visualize the surface $z = f(x, y)$.

Ⅱ. (3pt x 6 = 18) State or Define.

1. Choose 2 terminologies or concepts from each group (**A, B, C**) and state their meanings as much as you can.

A. ① Integral Test ② p-series Test ③ Comparison Test ④ Limit Comparison Test ⑤ Alternating Series Test ⑥ Ratio Test ⑦ Root Test

Theorem 1 The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words,

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 2 p-series Test

The series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Theorem 3 The Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series of *positive* terms with $a_n \le b_n$ for all $n \ge N$, where N is some fixed integer.
(i) If $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.
(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Theorem 4 The Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series of positive terms. Assume that

$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ exists. Then

$\sum a_n$ converges if and only if $\sum b_n$ converges.
$\sum a_n$ diverges if and only if $\sum b_n$ diverges.

Theorem 1 The Alternating Series Test

The alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \text{ where } b_n > 0 \text{ for all } n.$$

is convergent if it satisfies
(i) $b_{n+1} \le b_n$ for all n
(ii) $\lim_{n\to\infty} b_n = 0$.

Definition 2

A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

A series $\sum a_n$ is conditionally convergent if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Theorem 4 **The Ratio Test**

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn on the convergence or divergence of $\sum a_n$.

Theorem 5 **The Root Test**

(i) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

B. ⑧ Taylor series and Maclaurin series ⑨ Gamma function : gamma (2, 1) ⑩ Arc Length ⑪ Curvature
⑫ Limit of a function with two variables

Theorem 2 **Taylor's Theorem with Remainder**

Let f be n times continuously differentiable on the interval $|x-a| < h$. Then for all x in this interval,

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$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

where $R_n(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n$, for some point ξ lying between x and a.

The polynomial $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the nth **degree Taylor polynomial of** f **at** a, and $R_n(x)$ the **remainder** of the Taylor series of f about a. As an immediate corollary of the above theorem, we obtain the following important result.

Corollary 3

Let f be infinitely differentiable (smooth) on the interval $|x-a| < h$. Then f is equal to the sum of its Taylor series on the interval $|x-a| < R$ if and only if $\lim_{n\to\infty} R_n(x) = 0$, for $|x-a| < R$, where $R_n(x)$ is given by 3 in Theorem 2.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad (-1,\ 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad (-\infty,\ \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad (-\infty,\ \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad [-1,\ 1]$$

12.3 Arc Length and Curvature

Notice that for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$

and hence the arc length formula can be also be written as

$$L = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Curvature of Curves

Definitio

The **curvature** κ of a curve $\mathbf{r}$ is the magnitude of the rate of change of the unit tangent vector with respect to arc length. That is,

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

C. ⑬ Cross product of two vectors ⑭ Cauchy-Schwarz Inequality ⑮ Triangle Inequality ⑯ Projection

⑰ Tangential and normal components of the acceleration vector, $\mathbf{a}_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$ and $\mathbf{a}_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$

Definition1 Cross Product of Two Vectors

The cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

The cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ can also be obtained as follows

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

or in the determinant form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{array}{l} \leftarrow \text{Unit vectors} \\ \leftarrow \text{Components of } \mathbf{a}. \\ \leftarrow \text{Components of } \mathbf{b} \end{array}$$

Continuity of Functions of Three Variables

The notions of limit and continuity for functions of three or more variables are natural extensions of those just considered. For example, a function $w = f(x, y, z)$ is continuous at (a, b, c) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c).$$

...

Ⅲ. (2pt x 19 = 38pt) Find or Explain or Fill the blanks.

1. Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{e}{10}\right)^n$ is convergent or divergent. Find the sum if it is convergent.

▶▶ **Sol** This is a geometric series and since $-1 < \frac{e}{10} < 1$, this series is convergent.

And, $\sum_{n=1}^{\infty}\left(\frac{e}{10}\right)^n = \frac{\frac{e}{10}}{1-\frac{e}{10}} = \frac{e}{10-e}$.

http://matrix.skku.ac.kr/cal-lab/cal-10-1-19.html

```
var('n')
a(n) =  ((e)/10)^n
sum(a(n), n, 1, +oo)
```

Answer: -e/(e - 10) ■

2. Use $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, $|x|<1$ to find a power series representation for $f(x)=\ln(1-x^2)$. What is the radius of convergence?

▶▶ **Sol** $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$, $|x|<1$

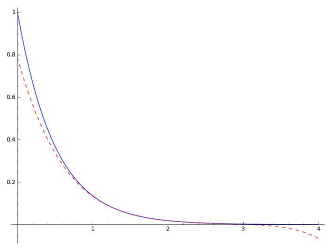
$\Rightarrow \ln(1-x^2) = -\sum_{n=1}^{\infty}\frac{(-x^2)^n}{n}$, $|x|<1$.

The radius of convergence R = 1 ■

3. Obtain the 5th Taylor polynomial of $f(x)$ around a, where $f(x) = e^{-2x}$, a=2

▶▶ **Sol** $T(x) = \sum_{n=0}^{5}\frac{f^{(n)}(2)}{n!}(x-2)^n = e^{-4} - 2e^{-4}(x-2) + 2e^{-4}(x-2)^2 - \frac{4e^{-4}}{3}(x-2)^3 + \frac{2e^{-4}}{3}(x-2)^4 - \frac{4e^{-4}}{15}(x-2)^5$.

```
var('x')
f(x)=exp(-2*x)
g(x)=f.taylor(x,2,5)
p1=plot(f,(x,0,4))
p2=plot(g,(x,0,4),color="red", linestyle='--')
show(p1+p2)
print g
```



x |--> -4/15*(x - 2)^5*e^(-4) + 2/3*(x - 2)^4*e^(-4) - 4/3*(x - 2)^3*e^(-4) + 2*(x - 2)^2*e^(-4) - 2*(x - 2)*e^(-4) + e^(-4) ■

4. Evaluate the indefinite integral $\int \frac{\sin t}{t}dt$ as an infinite series.

▶▶ **Sol** $\frac{\sin t}{t} = \frac{1}{t}(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots$

$\int \frac{\sin t}{t}dt = t - \frac{1}{3}\frac{t^3}{3!} + \frac{1}{5}\frac{t^5}{5!} - \cdots + C = (-1)^n\frac{t^{2n+1}}{(2n+1)\cdot(2n+1)!} + \cdots + C$ (C is constant, $\forall t$)

Also, we can get the result by Sage.

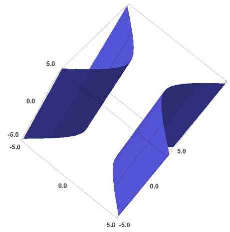
```
var('x')
f=sin(x)/x
g=f.taylor(x,0,10)
integral(g,x)
```

Answer : -1/439084800*x^11 + 1/3265920*x^9 - 1/35280*x^7 + 1/600*x^5 - 1/18*x^3 + x ■

5. Draw the surface of $x^2 - y^2 = 3$ in R^3.

Sol

```
var('x,y,z')
implicit_plot3d(x^2-y^2==3          , (x, -5, 5), (y, -5, 5), (z, -5, 5))
```



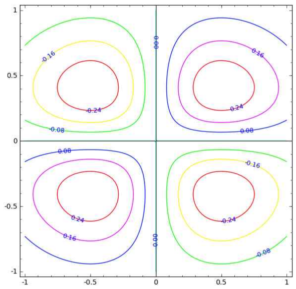
■

6. Draw the level curves(contour lines) for the function $f(x,y) = 4xye^{-2x^2-3y^2}$.

Sol

(Sage)

```
var('x, y')
f(x, y)=4*x*y*exp(-2*x^2-3*y^2)
contour_plot(f(x, y), (x, -1, 1), (y, -1, 1), fill=False, cmap='hsv', axes=True, labels=True)
```



7. Determine the dot product and cross product of two vectors $\mathbf{a} = \langle 6, -3, 4 \rangle$, $\mathbf{b} = \langle 2, 0, -3 \rangle$. Verify whether they are orthogonal, parallel, or neither.

Sol

```
a=vector([6,-3,4])
b=vector([2,0,-3])
c=a.dot_product(b)
d=a.cross_product(b)
print (c, d)
```

Answer : 0, <9, 26, 6>

They are orthogonal, . ■

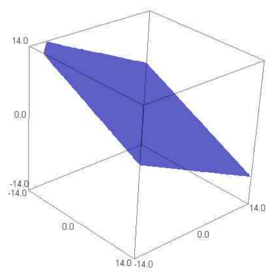
8. Find an equation of the plane through the point (2, -3, 4) and with normal vector (3, 3, 7), and find the distance from the point to the given plane.

Sol The equation of the plane through a point (x_0, y_0, z_0) with the normal vector $\mathbf{n} = (a, b, c)$ is given as:

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ when (x_0, y_0, z_0)=(2, -3, 4) and $\mathbf{n} = (3, 3, 7)$.

$\Rightarrow 3(x-2)+3(y+3)+7(z-4)=0 \qquad \therefore 3x+3y+7z=25$

```
n=vector(QQ, [3,3,7])
d=-25
p=vector(QQ, [2,-3,4])
dis=abs(n.dot_product(p)+d)/n.norm()
var('x, y, z')
P1=implicit_plot3d(3*x+3*y+7*z==25,(x,-14,14),(y,-14,14),(z,-14,14),color='blue',opacity=0.6)
show(P1)
print (dis)
```



Answer : : distance= 0 ■

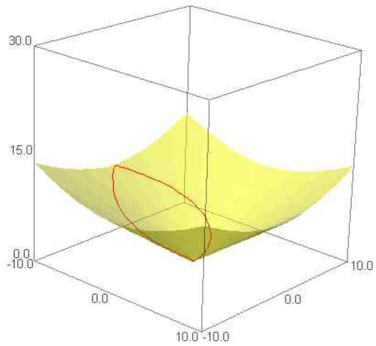
9. Show that the curve with parametric equation $x = t^2\cos t$, $y = t^2\sin t$, $z = t^2$ lies on the cone $z^2 = x^2 + y^2$.

http://matrix.skku.ac.kr/cal-lab/cal-13-1-12.html

▶▶ **Sol** We have $x^2 + y^2 = t^2\sin^2 t + t^2\cos^2 t = t^2 = z$.

```
var('x, y, z, t');
p1=parametric_plot3d((t^2*cos(t), t^2*sin(t), t^2), (t, -pi, pi), color='red'          );
p2=implicit_plot3d(z^2==x^2+y^2, (x, -10, 10), (y, -10, 10), (z, 0, 30), color='yellow',
opacity=0.5);
show(p1+p2);
```

■



10. Find the domain of the vector function $\mathbf{r}(t) = \ln(4-t^2)\mathbf{i} + \frac{t+1}{t-1}\mathbf{j} + \cos t\mathbf{k}$.

▶▶ **Sol** $4 - t^2 > 0$ and $t \neq 1$.

$\Rightarrow \{t \in R \mid -2 < t < 2 \text{ and } t \neq 1\}$. ■

11. Find $\mathbf{r}'(t)$ and draw the position vector $\mathbf{r}(t) = < \cos t, t >$ and the tangent vector $\mathbf{r}'(t)$ for the given value of $t = \frac{\pi}{4}$.

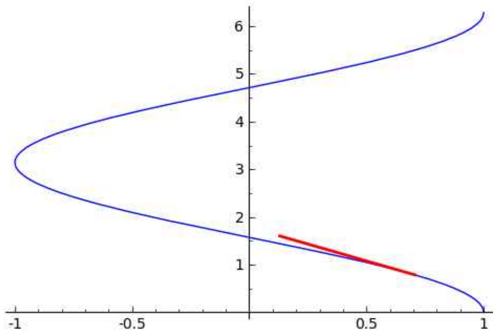
http://matrix.skku.ac.kr/cal-lab/cal-13-2-1.html

▶▶ **Sol**

```
var('t');
x(t)=cos(t);
y(t)=t;
r(t)=(x(t), y(t));
dr(t)=(diff(x(t), t)          , diff(y(t), t)          );
dr(t)
```

Answer : $\mathbf{r}'(t)$= < -sin(t), 1>

```
t0=pi/4;
p1=r(t0);
p2=dr(t0)/abs(dr(t0))+r(t0);
p=parametric_plot((x(t), y(t)), (t, 0, 2*pi));
utv=line([p1, p2], rgbcolor=(1, 0, 0), width=2);
show( p + utv );
```



■

12. Find the length of the curve $\mathbf{r}(t) = < 4t, 3\cos t, 3\sin t >$, $-5 \leq t \leq 5$.

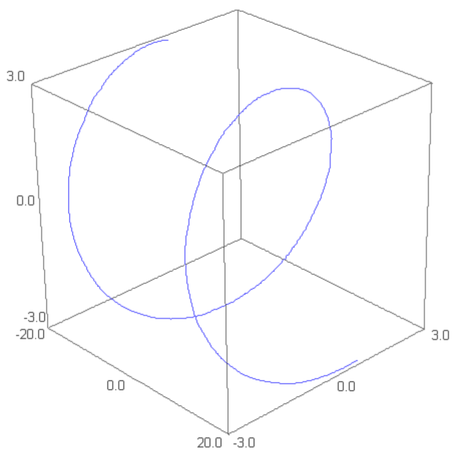
▶▶ **Sol** Find $\mathbf{r}'(t)$ and $s(t) = |\mathbf{r}'(t)|$ and $L = \int_{-5}^{5} s(t)dt$.

Using Sage)

```
var('t')
r(t)=(  4*t, 3*cos(t), 3*sin(t)        )
parametric_plot3d(r(t),(t,-5,5)); show(p)

dr=diff(r(t),t)
s(t)=sqrt((   dr[0]^2+dr[1]^2+dr[2]^2    ).simplify_trig());
length=integral(s(t), t, -5, 5)
length
```

Answer : 50 ■



13. Find the unit tangent $\mathbf{T}(t)$, unit normal vectors $\mathbf{N}(t)$ and the curvature κ when $\mathbf{r}(t) = < 5\cos t, 5\sin t, 4t >$.

▶▶ **Sol** $\mathbf{r}'(t) = <-5\sin t,\ 5\cos t,\ 4>$, $|\mathbf{r}'(t)| = \sqrt{(-5\sin t)^2 + (5\cos t)^2 + 4^2} = \sqrt{41}$.

$\Rightarrow$ **the unit tangent** $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle -\dfrac{5}{\sqrt{41}}\sin t,\ \dfrac{5}{\sqrt{41}}\cos t,\ 4 \right\rangle$.

$\Rightarrow$ $\mathbf{T}'(t) = \left\langle -\dfrac{5}{\sqrt{41}}\cos t,\ -\dfrac{5}{\sqrt{41}}\sin t,\ 0 \right\rangle$ and $|\mathbf{T}'(t)| = \dfrac{5}{\sqrt{41}}$.

Hence **the unit normal** vectors $= \mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = <-\cos t,\ -\sin t,\ 0>$.

$\Rightarrow$ **the curvature** $\kappa = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{5/\sqrt{41}}{\sqrt{41}} = \dfrac{5}{41}$.

Using Sage multi-cell) http://math1.skku.ac.kr/home/pub/1362/

Cell NO.	1

Code	var('t')
	r(t)=(5*cos(t),5*sin(t),4*t)
	dr=diff(r(t),t)
	s(t)=sqrt((dr[0]^2+dr[1]^2+dr[2]^2).simplify_trig()); #dr.norm() 도 OK
	show(s(t))

Answer	$\sqrt{41}$

Cell NO.	2
Code	T(t)=(dr[0]/s(t),dr[1]/s(t),dr[2]/s(t)) show(T(t))
Answer	$(-541\sqrt{41}\sin(t),\ 541\sqrt{41}\cos(t),\ 441\sqrt{41})$

Cell NO.	3
Code	dT=diff(T(t),t) d(t)=sqrt((dT[0]^2+dT[1]^2+dT[2]^2).simplify_trig()); show(d(t))
Answer	$5\sqrt{141}$

Cell NO.	4
Code	N(t)=(dT[0]/d(t),dT[1]/d(t),dT[2]/d(t)) show(N(t))
Answer	$(-\sqrt{141}\ \sqrt{41}\cos(t),\ -\sqrt{141}\ \sqrt{41}\sin(t),0)$

Cell NO.	5
Code	k=d(t)/s(t) k.simplify_full()
Answer	5/41

■

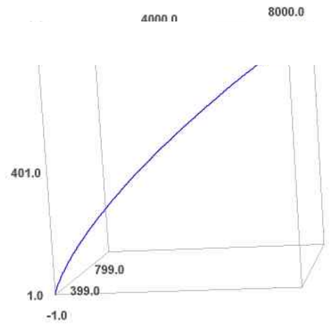
14. Find the velocity, acceleration and speed of a particle with the given position function $\mathbf{r}(t) = <2t^2+1,\ t^3,\ 2t^2-1>$.

▶▶ **Sol** $\mathbf{v}(t) = \mathbf{r}'(t) = <4t,\ 3t^2,\ 4t>$

$\Rightarrow$ $\mathbf{a}(t) = \mathbf{r}''(t) = <4,\ 6t,\ 4>$ $\Rightarrow$ $s(t) = |\mathbf{v}(t)| = \sqrt{(4t)^2 + (3t^2)^2 + (4t)^2} = \sqrt{9t^4 + 32t^2}$

http://math1.skku.ac.kr/home/pub/1355/

```
r(t)= (2*t^2 + 1, t^3, 2*t^2 - 1)
v= diff(r(t), t)            #v는 속도벡터함수, diff는 미분하는 명령어
a= diff(v(t), t)            #a는 가속도벡터함수
s= v.norm()                 #s는 속도의 크기, .norm()은 벡터의 크기를 구한다.
p1 = parametric_plot3d(r(t), (t, 0, 20), color='blue', width= 2)
show (p1)
r(t); v; a; s
```



Position Vector : (2*t^2 + 1, t^3, 2*t^2 - 1)
Velocity Vector : (4*t, 3*t^2, 4*t)
Acceleration Vector : (4, 6*t, 4)
Speed : sqrt(abs(3*t^2)^2 + 2*abs(4*t)^2) ■

15. The position function of a particle is given by $\mathbf{r}(t) = < 2t,\ t,\ t^2 - 4t >$. When is the speed a minimum?

Sol $\mathbf{v}(t) = \mathbf{r}'(t) = < 2, 1, 2t - 4 >$

$$s(t) = |\mathbf{v}(t)| = \sqrt{2^2 + 1^2 + (2t-4)^2} = \sqrt{4(t-2)^2 + 5}$$

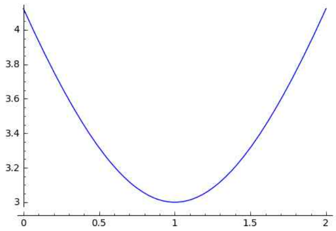
$\therefore$ When $t = 2$, the speed is minimum.

http://math1.skku.ac.kr/home/pub/1358/

```
var('t')
r(t)=(2*t, t, t^2-4*t)
v=diff(r(t), t)
s=v.norm()
s0=diff( s    , t)
solve(s0==0, t)
```

[t==2]

```
plot (s, (t, 0, 2))
```



Answer : $t = 2$ ■

16. Find the projection $\text{proj}_{\mathbf{x}}\mathbf{y}$ of $\mathbf{y}$ onto $\mathbf{x}$ and a vector component of $\mathbf{y}$ perpendicular to $\mathbf{x}$ for $\mathbf{x} = (4, -1, 3)$ and $\mathbf{y} = (3, 1, -2)$.

Sol Since $\mathbf{y} \cdot \mathbf{x} = 5$ and $|\mathbf{x}| = \sqrt{26}$, the projection of $\mathbf{y}$ onto $\mathbf{x}$ is

$$\text{proj}_{\mathbf{x}}\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}\mathbf{x} = \frac{5}{26}(4, -1, 3) = \left(\frac{10}{13}, -\frac{5}{26}, \frac{15}{26}\right)$$ and a vector component of $\mathbf{y}$ perpendicular to $\mathbf{x}$ is

$$\mathbf{w} = \mathbf{y} - \text{proj}_{\mathbf{x}}\mathbf{y} = (3, 1, -2) - \left(\frac{10}{13}, -\frac{5}{26}, \frac{15}{26}\right)$$

$$= \left(\frac{29}{13}, \frac{31}{26}, -\frac{67}{26}\right).$$ ■

17. Compute $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $2\mathbf{a}-3\mathbf{b}$ and $|\mathbf{a}-\mathbf{b}|$ where $\mathbf{a} = \langle 5, -1, 3 \rangle$, $\mathbf{b} = \langle -1, 3, -2 \rangle$.

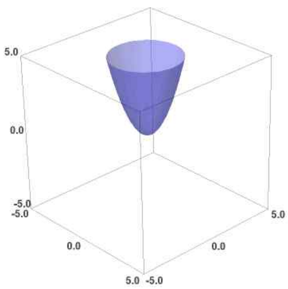
Sol $|\mathbf{a}| = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{35}$

$\mathbf{a}+\mathbf{b} = \langle 5, -1, 3 \rangle + \langle -1, 3, -2 \rangle = \langle 4, 2, 1 \rangle$

$2\mathbf{a} - 3\mathbf{b} = 2\langle 5, -1, 3 \rangle - 3\langle -1, 3, -2 \rangle = \langle 10, -2, 6 \rangle - \langle -3, 9, -6 \rangle = \langle 13, -11, 12 \rangle$

$|\mathbf{a}-\mathbf{b}| = |\langle 5, -1, 3 \rangle - \langle -1, 3, -2 \rangle| = |\langle 6, -4, 5 \rangle| = \sqrt{6^2 + (-4)^2 + 5^2} = \sqrt{77}$

18. Describe and sketch an equation for the surface obtained by rotating the parabola $z = x^2$ about the z-axis.



Sol 포물선 $z = x^2$을 z축을 축으로 회전하면, 옆의 그림과 같이 $z = c$ 평면 위에서 원들로 나타나므로, $z = x^2 + y^2$ 라는 포물면의 방정식을 얻는다.

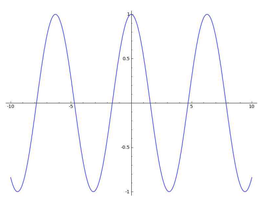
```
var('x,y,z')
implicit_plot3d(z==x^2+y^2, (x, -5, 5), (y, -5, 5), (z, -5, 5), opacity=0.5)
```

Answer : The equation is $z = x^2 + y^2$ ■

19. Describe and sketch the surface $z = \cos y$.

http://matrix.skku.ac.kr/cal-lab/cal-11-6-4.html

Sol yz축에서 $z = \cos y$ 을 그린 후, $x-$축을 따라 extend 한다.



z=cos y

```
var('x, y')
z=cos(y)
plot3d(z, (x, -2, 2), (y, -pi, pi))
```

■

IV. (3+5= 8pt) Prove or Explain (Fill the blanks).

1. Show $\lim_{n\to\infty}\frac{2n-5}{3n}=\frac{2}{3}$ using ($\epsilon-\delta$ or) $\epsilon-N$ method.

▸ ▹ Proof : $\forall\ \epsilon>0$ [Find N (depending on ε)] so that

■ [Side Calculation] $\left|\frac{2n-5}{3n}-\frac{2}{3}\right|=\left|\frac{2n-5}{3n}-\frac{2n}{3n}\right|=\left|\frac{5}{3n}\right|<\epsilon$ for every $n\geq N$. If $n>\frac{5}{3\epsilon}\Rightarrow\frac{5}{3n}<\epsilon$.

Let N be any integer larger than $\frac{5}{3\epsilon}$ (follows from Archimedean Property).

$\Rightarrow$ Then we have $\frac{5}{3n}<\epsilon$ for all $n\geq N$, as desired. ■

2. Explain that the limit

$$\lim_{(x,y)\to(0,0)}\frac{x^3y}{x^6+y^2}$$

does NOT exist.

▸▸ **Sol** Let $x\to0$ and $y\to0$ in such a way that $y=mx$. Then along this line

$$\lim_{(x,y)\to(0,0)}\frac{x^3y}{x^6+y^2}=\lim_{x\to0}\frac{mx^4}{x^6+m^2x^2}=\lim_{x\to0}\frac{mx^2}{x^4+m^2}=\frac{0}{0+m^2}=0$$

However, let $x\to0$ and $y\to0$ in such a way that $y=mx^3$. Then along this curve

$$\lim_{(x,y)\to(0,0)}\frac{x^3y}{x^6+y^2}=\lim_{x\to0}\frac{mx^6}{x^6+m^2x^6}=\lim_{x\to0}\frac{mx^6}{(1+m^2)x^6}=\frac{m}{1+m^2}$$

Since the limit of the function depends on the manner in which (x,y) approaches to $(0,0)$, the function $f(x,y)=\frac{x^3y}{x^6+y^2}$ does not have a limit at $(0,0)$. For example, for $m=1$, $f(x,y)\to\frac{1}{2}$. ■

****

Students presentation:

? 미적분학 with Sage Sec-9-1 Sequences and Series, 문제풀이 by 이원준 http://youtu.be/O6y1v5fJA0k
? 미적분학 with Sage Sec-9-2 Tests for convergence of series with positive terms, 문제풀이 by 김범윤 http://youtu.be/1flKAnlv9LA
*미적분학 with Sage Sec-9.3 Calculus of Alternating Series and Absolute Convergence, 문제풀이 by 김유경 http://youtu.be/XESJ9Hg_qbo

*미적분학 with Sage Sec-9.4 Calculus of Power Series, 문제풀이 by 이한울 http://youtu.be/zPnbn6n3b1w
*미적분학 with Sage Sec-9.5 Calculus of Taylor, Maclaurin, and Binomial Series, 문제풀이 by 이지석 http://youtu.be/IDniIOPtNaE

*미적분학 with Sage Sec-11.1 Three-Dimensional Coordinate Systems, 문제풀이 by 김태현 http://youtu.be/_s_2T1VVob8
? 미적분학 with Sage Sec-11.2 Vectors, 문제풀이 by 오교혁 http://youtu.be/BFgh6irMqsc
*미적분학 with Sage Sec-11.3 Calculus of The Dot Product, 문제풀이 by 서용태 http://youtu.be/qKTqcazFEhw
? 미적분학 with Sage Sec-11.4 Calculus of The Vector or Cross Product, 문제풀이 by 최주영 http://youtu.be/0gtL8BqjU2c
*미적분학 with Sage Sec- 11.5 Equations of Lines and Planes, 문제풀이 by 구본우 http://youtu.be/lxuGE_Erthg
*미적분학 with Sage Sec- 11.6 : 1조 조장 김건호 군

? PDF : 미적분학 with Sage Sec-12.1 Vector-Valued Functions and Space Curves, 문제풀이 by 최양현 http://youtu.be/jvMI6OzdR_I
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*미적분학 with Sage Sec-12.3 Arc Length and Curvature, 문제풀이 by 이한울 http://youtu.be/w-R4ez7vg24
*미적분학 with Sage Sec-12.4 Velocity and Acceleration, 문제풀이 by 이인행 http://youtu.be/JlgB5XN3LI4

Calculus Map by SGLee