Spring 2016, LA Midterm Exam - 채점용 답안(1 hour In class Exam)						Sign
Course	Linear Algebra	GEDB003	41	Prof.	Sang-Gu Lee	
Major		Year 학년		Student No. 학번		Name

※ Notice

1. Fill out the above boxes before you start this Exam. (학번, 이름 등을 기입하고 감독자 날인)
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4. You may use the following <Sage codes> in your answers. (중간고사까지는 한국어 답안도 OK)

Total Score (100 pt)	
Offline Exam 84	Participation 16
84	16

```
var('a, b, c, d')                 # Define variables
eq1=3*a+3*b==12                   # Define equation1
eq2=5*a+2*b==13                   # Define equation2
solve([eq1, eq2], a,b)            # Solve eq's
A=matrix(QQ, 3, 3, [3, 0, 0, 0, 0, 2, 0, 3, 4]);     # Matrix
x=vector([3, 1, 2])               # Define vector x
A.augment(x)                      #  [A: x]
A.echelon_form()                  # Find RREF
A.inverse()                       # Find  inverse
A.det()                           # Find determinant
A.adjoint()                       # Find adjoint matrix
A.charpoly()                      # Find charct. ploy
A.eigenvalues()                   # Find eigenvalues
A.eigenvectors_right()            # Find eigenvectors
A.rank()                          # Find rank of A
A.right_nullity()                 # Find nullity of A
var('t')                          # Define variables
x=2+2*t                           # Define a parametric eq.
y=-3*t-2
bool( A== B)                      # Are A and B same?
```

```
var('x, y')                                  # Define variables
f = 7*x^2 + 4*x*y + 4*y^2-23                 # Define a function
implicit_plot( f, (x, -10, 10), (y, -10, 10))   # implicit Plot
parametric_plot((x,y), (t, -10, 10), rgbcolor='red')  # Plot
plot3d(y^2+1-x^3-x, (x, -pi, pi), (y, -pi, pi))      # 3D Plot
A=random_matrix(QQ,7,7)   # random matrix of size 7 over Q
F=random_matrix(RDF,7,7)  # random matrix of size 7 over R
P,L,U=A.LU()              # LU  (P: Permutation M. / L, U
print P, L, U

h(x, y, z) = [x+2*y-z, y+z, x+y-2*z]
T = linear_transformation(U, U, h)   # L.T.
print T.kernel()          # Find a basis for kernel(T)
C=column_matrix([x1, x2, x3])
D=column_matrix([y1, y2, y3])
aug=D.augment(C, subdivide=True)
Q=aug.rref()

[G,mu]=A.gram_schmidt()     # G-S
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B #

A.H  # conjugate transpose of A
A.jordan_form()   # Jordan Canonical Form of A
```

<Sample Sage Linear Algebra codes>

I. (1pt x 20= 20pt) True(T) or False(F).

1. (F) For any two $n\times n$ matrices A and B, $\det(AB)=\det(B)\det(A)$ and $(AB)^T=A^TB^T$
2. (T) A given matrix can be written uniquely as a sum of a symmetric matrix and a skew-symmetric matrix.
3. (T) If $A=P^{-1}BP$, then $\det A=\det B$.
4. (T) If $A, B\in M_n$ are lower triangular matrices, then the product $C=AB$ is also lower triangular.
5. (F) Every linear independent vectors in $\mathbb{R}^n$ are orthogonal vectors.
6. (T) Any subspace of $\mathbb{R}^2$ is either one of a line through the origin or $\mathbb{R}^2$ or $\{0\}$.
7. (T) For a set of natural numbers $S=\{1, 2, \ldots, n\}$, permutation is a one to one function from S to S.
8. (T) The set $S=\{(1,1,0),(0,0,-2),(0,1,3)\}$ spans $\mathbb{R}^3$.
9. (T) Finding all solutions of a polynomial is a same problem of finding all eigenvalues of a corresponding (companion) matrix.
10. (T) There is a bijective function (one to one correspondence) between the set of all permutations on $S_n=\{1,2,\ldots,n\}$ and the set of all $n\times n$ permutation matrices.
11. (T) If A is a $n\times n$ real orthogonal matrix, then the linear mapping $\mathbf{x}\mapsto A\mathbf{x}$ preserves length.
12. (T) For a transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, if $T(\mathbf{u})=T(\mathbf{v}) \Rightarrow \mathbf{u}=\mathbf{v}$, then it is called injective.
13. (T) For a nonzero vector $\mathbf{n}=(a_1,a_2,\cdots,a_n)\in\mathbb{R}^n$, the hyperplane $\Pi:\mathbf{n}^\perp=\{\mathbf{x}\in\mathbb{R}^n\mid \mathbf{n}\cdot\mathbf{x}=0,\ \mathbf{x}\in\mathbb{R}^n\}$ is called the orthogonal complement of $\mathbf{n}$. And the vector $\mathbf{n}$ is called the normal vector of the hyperplane Π.
14. (T) A normal vector perpendicular to the plane $z=3y+8$ is $\mathbf{n}=(0,3,-1)$.
15. (T) If E is an elementary matrix, then A and EA has a same solution set.
16. (T) If there are n unknowns and k free variables in $A\mathbf{x}=\mathbf{b}$, then there are $n-k$ leading variables.
17. (F) Every vector in the eigenspace of A corresponding to its eigenvalue λ is an eigenvector.
18. (F) For any $n\times n$ matrix A with $n>1$, $\det(\operatorname{adj}A)=(\det A)^n$.
19. (T) Let $A=[a_{ij}]_{n\times n}\in M_n(\mathbb{R})$. If $T_A:\mathbb{R}^n\to\mathbb{R}^n$ is a linear transformation, T_A is one-to-one if and only if T_A is onto.
20. (F) For linear transformations $T:\mathbb{R}^n\to\mathbb{R}^k$ and $S:\mathbb{R}^k\to\mathbb{R}^m$, $S\circ T$ is one-to-one implies S is one-to-one.

II. (2pt x 5 = 10pt) State or Define (Choose/Mark 5 only: Fill the boxes and/or state).

1. State the Cauchy-Schwarz inequality

For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$$

Th equality holds iff $\mathbf{x} = k\mathbf{y}$ for some $k \in \mathbb{R}$. ■

2. [Laplace cofactor expansion] Let A be an $n \times n$ matrix. For any i, j $(1 \leq i, j \leq n)$ the following holds.

$|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$ (cofactor expansion along the 1st row)

■

3. [Determinant] The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is defined as

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

■

4. [Linearly independent, linearly dependent]

If $S = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\} \subseteq \mathbb{R}^n$ satisfies

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = 0 \quad (c_1, c_2, \ldots, c_k \in \mathbb{R})$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

then $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ (or subset S) are called **linearly independent.**

If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ (or subset S) are not linearly independent, then it is called **linearly dependent**. ■

5. [kernel] Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

The kernel of T = $\ker T = \{\mathbf{v} \in \mathbb{R}^n \mid T(\mathbf{v}) = 0\}$.

■

6. [(Real) orthogonal matrix] An $n \times n$ real matrix $A = [a_{ij}]$ is called an orthogonal matrix if

$$A^{-1} = A^T$$ ■

7. [Standard Matrix of L.T.]

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation where $T(\mathbf{x}) = A\mathbf{x}$, $\forall \mathbf{x} \in \mathbb{R}^n$, then the standard matrix $A = [T]$ of T is

$$A = [\,T(\mathbf{e}_1) : T(\mathbf{e}_2) : \cdots : T(\mathbf{e}_n)\,].$$

■

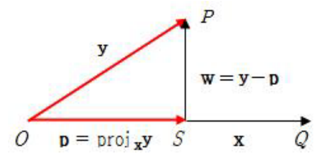
8. [skew-symmetric matrix] An $n \times n$ real matrix $A = [a_{ij}]$ is called a skew-symmetric matrix if

$$A^T = -A$$

■

III. (3pt x 8 = 24pts) Find or Explain (Fill the boxes) :

1. [$\text{proj}_{\mathbf{x}}\mathbf{y}$] For vectors $\mathbf{y} = (3, -1, 2)$ and $\mathbf{x} = (-4, 4, 2)$, find the vector $\mathbf{w} = \mathbf{y} - \text{proj}_{\mathbf{x}}\mathbf{y}$ in $\mathbb{R}^3$ and $\mathbf{y} \cdot \mathbf{w}$.



Sol $\text{proj}_{\mathbf{x}}\mathbf{y} = \dfrac{\mathbf{y} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}\mathbf{x} = \left(\frac{4}{3}, -\frac{4}{3}, -\frac{2}{3}\right)$, $\mathbf{w} = \mathbf{y} - \text{proj}_{\mathbf{x}}\mathbf{y} = \left(\frac{5}{3}, \frac{1}{3}, \frac{8}{3}\right)$. $\mathbf{y} \cdot \mathbf{w} = 10$. ■

2. Find vector equation of the plane that passes a line $x = 1 + 3t,\ y = 1 - 2t,\ z = -2 + 2t\ (t \in \mathbb{R})$ and one point $P(4,1,1)$.

Sol $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (t_1, t_2 \in \mathbb{R})$

On l : $x = 1 + 3t,\ y = 1 - 2t,\ z = -2 + 2t$ for all $t \in \mathbb{R}$.

Say $Q(4, -1, 0)$ when $t = 1$. Say $R(1, 1, -2)$ when $t = 0$.

$\Rightarrow \mathbf{v}_1 = \overrightarrow{PQ} = (0, -2, -1)$, $\mathbf{v}_2 = \overrightarrow{PR} = (-3, 0, -3)$, $\overrightarrow{OP} = \mathbf{x}_0 = (4,1,1)$

$$\Rightarrow \mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 = (4,1,1) + t_1(0, -2, -1) + t_2(-3, 0, -3)$$
$$= (4 - 3t_2,\ 1 - 2t_1,\ 1 - t_1 - 3t_2)$$

$\therefore$ $\mathbf{x} = (4 - 3t_2,\ 1 - 2t_1,\ 1 - t_1 - 3t_2)$ is the vector equation of the plane. ■

3. (5pts) Let A be a 3×3 matrix, and assume that $\text{adj}\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 7 & 6 \end{bmatrix}$

(1) Find $\det(\text{adj}\,A)$.

(2) Find A.

Sol

(1) $\det(\text{adj}\,A) = (\det(A))^{n-1}$ $\quad\therefore\quad$ $\det(\text{adj}\,A) = 4$,

(2) From $A^{-1} = \dfrac{1}{|A|}\text{adj}\,A$, $A = (A^{-1})^{-1} = \left(\dfrac{1}{|A|}\text{adj}\,A\right)^{-1} = |A|(\text{adj}\,A)^{-1}$, $(\det(A))^2 = \det(\text{adj}\,A)$.

$$A = \pm 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 7 & 6 \end{bmatrix}^{-1} = \pm 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{7}{4} & \frac{3}{4} \end{bmatrix} = \pm \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -\frac{7}{2} & \frac{3}{2} \end{bmatrix}$$

■

4. Let T_1 and T_2 are defined as follows:

$$T_1(x_1, x_2, x_3) = (4x_1, -2x_1 + x_2, -x_1 - 3x_2), \quad T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3).$$

(1) Find the standard matrix for each T_1 and T_2.

(2) Find the standard matrix for each $T_2 \circ T_1$ and $T_1 \circ T_2$.

Sol

(1) $T_1\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$, $T_1\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, $T_1\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\quad\therefore [T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$

$T_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $T_2\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $T_2\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ $\quad\therefore [T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$

(2) $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$, $\quad [T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$ ■

Sage) http://math3.skku.ac.kr/home/pub/29

5. Let the characteristic polynomial of matrix A be $p(\lambda)=(\lambda-2)(\lambda-3)(\lambda-4)$. Find eigenvalues of matrix A^2.

Sol $p(\lambda)=(\lambda-2)(\lambda-3)(\lambda-4)$

=> $\lambda_1=2,\ \lambda_2=3,\ \lambda_3=4$ are eigenvalues of matrix A.

=> $A\mathrm{x}=\lambda\mathrm{x}$, so $A^2\mathrm{x}=\lambda A\mathrm{x}=\lambda(\lambda\mathrm{x})=\lambda^2\mathrm{x}$.

∴ The eigenvalues of matrix A^2 is λ^2, which are $\lambda_1^2=4$, $\lambda_2^2=9$ and $\lambda_3^2=16$. ■

6. Find a if $\lambda=1$ is an eigenvalue of $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & a \end{bmatrix}$.

Sol

$$|I-A|=\begin{vmatrix} -1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1-a \end{vmatrix}=4-4(1-a)=4a=0 \quad => \quad a=0.$$ ■

7. [Invertible Matrix Theorem] Let A be an $n\times n$ matrix.

Which of the following statements is not equivalent to "the matrix A is invertible."? (Choose two)

(1) Column vectors of A are linearly independent.
(2) Row vectors of A are linearly dependent.
(3) $A\mathrm{x}=\mathbf{0}$ has a unique solution $\mathrm{x}=\mathbf{0}$.
(4) For any $n\times 1$ vector b, $A\mathrm{x}=\mathrm{b}$ has a unique solution.
(5) A and I_n are row equivalent.
(6) A and I_n are column equivalent.
(7) $\det(A)=0$
(8) $\lambda=0$ is not an eigenvalue of A.
(9) $T_A: \mathbb{R}^n\to\mathbb{R}^n$ by $T_A(\mathrm{x})=A\mathrm{x}$ is one-to-one.
(10) $T_A: \mathbb{R}^n\to\mathbb{R}^n$ by $T_A(\mathrm{x})=A\mathrm{x}$ is onto.

Ans 2 , 7 . ■

8. Determine the number of free variables in the solution set (space) of the homogeneous linear system of equations.

$$\begin{cases} x-6y-2z-2w=0 \\ 3y+z+w=0 \\ 5x-18y-6z-6w=0 \end{cases}$$

Ans $A=\begin{bmatrix} 1 & -6 & -2 & -2 \\ 0 & 3 & 1 & 1 \\ 5 & -18 & -6 & -6 \end{bmatrix}$, $\mathrm{x}=\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$, $\mathrm{b}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\mathrm{RREF}(A^T)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

=> The matrix A^T has two linearly independent row vectors.

=> The matrix A has two linearly independent column vectors.

So the number of free variables in the solution set of the lhomogeneous linear system of equations is two. ■

IV. (5+5+5=15pt) Python/ Sage Computations.

1. (5pts) We solved the following LSE using Gauss-Jordan elimination (and Sage). Fill the blanks.

$$\begin{aligned} 2x+7y+z-2u+3v &= 1 \\ 3x-2y+4z-6u &= -2 \\ x+y-z-3u-v &= -1 \\ 5x-y+2z-8u+2v &= 3 \\ y+z+u+v &= 4 \end{aligned}$$

Sol Find RREF of [A : b].

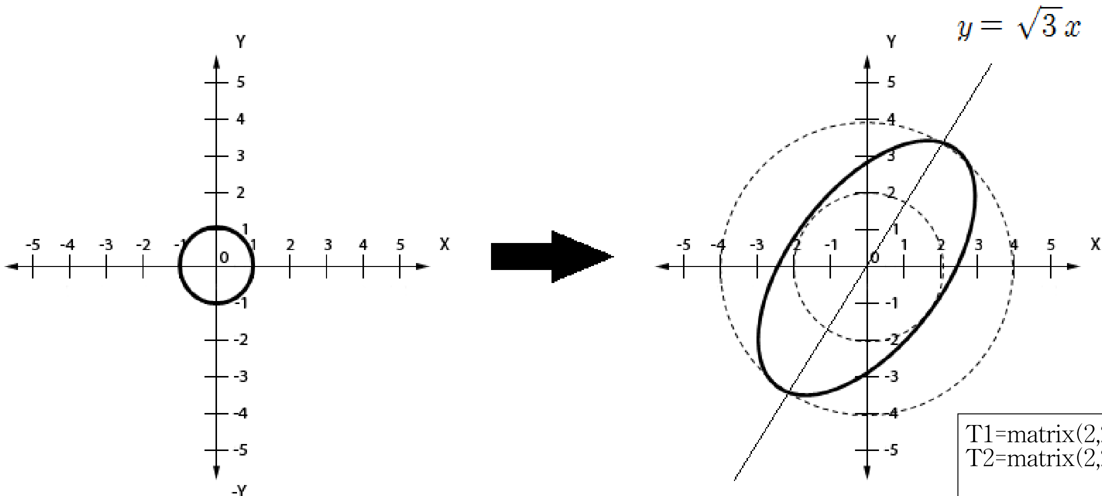
```
A=matrix([[2, 7, 1, -2, 3], [3, -2, 4, -6, 0], [1, 1, -1, -3, -1], [5, -1, 2, -8, 2], [0, 1, 1, 1, 1]])
b=vector([1, -2, -1, 3, 4])
print "RREF([A : b]) ="

print   A.augment( b ).rref()
```

실행(Evaluate)

```
RREF([A : b]) =
[     1      0      0      0      0   424/13]
[     0      1      0      0      0    20/13]
[     0      0      1      0      0    17/52]
[     0      0      0      1      0   425/26]
[     0      0      0      0      1  -739/52]
```

∴ The solution is $x=424/13$, $y=20/13$, $z=17/52$, $u=425/26$, $v=-739/52$. ■

2. (5pts) As shown in the picture, find a matrix transformation which transforms a circle with radius 1 to the given ellipse.



$[T_1]=\begin{bmatrix}4&0\\0&2\end{bmatrix}$, $[T_2]=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}$

∴ $[T_2][T_1]=\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}4&0\\0&2\end{bmatrix}=\begin{bmatrix}2&-\sqrt{3}\\2\sqrt{3}&1\end{bmatrix}$ ■

Let x= $\begin{bmatrix}x\\y\end{bmatrix}$ such that $x^2+y^2=1$.

∴ $(T_2\circ T_1)(\mathrm{x})=[T_2][T_1]\mathrm{x}=\begin{bmatrix}2x-\sqrt{3}y\\2\sqrt{3}x+y\end{bmatrix}$ ■

```
T1=matrix(2,2,[4,0,0,2]);
T2=matrix(2,2,[cos(pi/3), -sin(pi/3), sin(pi/3), cos(pi/3)]);

T=T2*T1    # Standard matrix for the given transformation.

var('x')  # Declare a variable.
var('y')  # Declare a variable.

Circle=matrix(2,1,[x, y]);
Ellipse=T*Circle

print T
print
print Ellipse
```

```
[         2   -sqrt(3)]
[2*sqrt(3)           1]    : OK ■

[-sqrt(3)*y + 2*x]
[ 2*sqrt(3)*x + y]         : OK ■
```

Sage : http://math3.skku.ac.kr/home/pub/343

3. (5pts) Consider $A\mathbf{x}=\mathbf{y}$ where $A=\begin{bmatrix} 7 & -2 & -2 & 1 & 0 \\ 3 & 0 & -2 & 1 & 2 \\ 12 & -4 & -3 & 2 & 0 \\ 6 & -8 & -4 & 6 & 4 \\ 1 & -2 & -2 & 1 & 6 \end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \\ 1 \end{bmatrix}$. You were asked to find

(1) Augment matrix [A: y] (2) RREF(A) (3) Det$A$ (4) Inverse of A (4) characteristic polynomial of A (5) all eigenvalues of A (6) all eigenvectors of A. The following is your answer. Fill out the blanks to find each.

Sol)

1) Step 1: Browse http://math3.skku.ac.kr or http://math1.skku.ac.kr/ (or http://sage.skku.edu/ or https://cloud.sagemath.com etc)
2) Step 2: Type class/your ID: ( ) and PW : ( )
3) Step 3: Click "New worksheet (새 워크시트)" button.
4) Step 4: Define a matrix A in the first cell in rational (QQ) field.

A = matrix(QQ,5,5,[7,-2,-2,1,0,3,0,-2,1,2,12,-4,-3,2,0,6,-8,-4,6,4,1,-2,-2,1,6]) and y = matrix(QQ,5,1,[0,2,4,2,1])

5) Step 5: Type a command to find augment matrix **[A: y]** A.augment(y) and evaluate
6) Step 6: Type a command to find **RREF**(A) (A.augment(y).rref()) and evaluate.
7) Step 7: Type a command to find **determinant of** A (**det(A)**) and evaluate.
8) Step 8: Type a command to find inverse of A (**A.inverse()**) and evaluate.
9) Step 9: Type a command to find char. polynomial of A (**A.charpoly()**) and evaluate.
10) Step 10: Type a command to find **eigenvalues** of A (**A.eigenvalues()**) and evaluate.
11) Step 11: Type a command to find **eigenvectors** of A (**A.eigenvectors_right()**) and evaluate.
13) Last step : Give 'print' command to see what you like to read.

Now we have some out from the Sage.

RREF(A) = Identity matrix of size 5

det(A) = 144

inverse(A) =

```
[   -1   1/6   2/3 -1/12     0]
[ -4/3   2/3   2/3 -1/12  -1/6]
[   -4   1/3   7/3  -1/6     0]
[ -8/3   5/6   4/3  1/12  -1/3]
[ -7/6   1/6   2/3 -1/12   1/6]
```

characteristic polynomial of (A) = x^5 - 16*x^4 + 95*x^3 - 260*x^2 + 324*x - 144

eigenvalues of A = { 6 , 4 , 3 , 2 , 1 } ■

eigenvectors = [(6, [(0, 1, 0, 2, 2)], 1), (4, [(1, 1, 2, 3, 1)], 1), (3, [(1, 1, 2, 2, 1)], 1), (2, [(0, 1, 0, 2, 0)], 1), (1, [(1, 1, 3, 2, 1)], 1)]

Write what (4, [(1, 1, 2, 3, 1)] , 1) means in eigenvectors of A :

[Answer] 4 is an eigenvalue of A and
its (algebraic) multiplicity is 1,
its corresponding eigenvector is (1, 1, 2, 3, 1),
and its geometric multiplicity is 1 . ■

V. (3pt x 5 = 15pt) Explain or give a sketch of proof.

1. (3pt) Explain how you are going to show whether $v_1 = (4, -5, 6)$, $v_2 = (-2, 1, 3)$, $v_3 = (6, -3, 10)$ are linearly independent or not.

Sol

(1) Make a 3 by 3 matrix and find the determinant of it.

(2) If it is not zero, they are linearly independent.

(3) If it is zero, they are linearly dependent. ■

2. Let A and I be $n \times n$ matrices. If $A+I$ is invertible, show that $A(A+I)^{-1} = (A+I)^{-1}A$.

Proof

$A^2 + A = (A+I)A = A(A+I)$

=> $(A+I)A(A+I)^{-1}(A+I) = (A+I)(A+I)^{-1}A(A+I)$

=> $(A+I)^{-1}(A+I)A(A+I)^{-1}(A+I)(A+I)^{-1} = (A+I)^{-1}(A+I)(A+I)^{-1}A(A+I)(A+I)^{-1}$

=> $A(A+I)^{-1} = (A+I)^{-1}A$ ■

3. Explain why there are no matrices $A, B \in M_n$ that satisfies $AB - BA = I_n$.
(Hint: Compare the trace of $\mathrm{tr}(AB-BA)$ and $\mathrm{tr}(I_n)$)

Proof

BWOC Suppose there are matrices $A, B \in M_n$ that satisfies $AB - BA = I_n$

=> $\mathrm{tr}(AB-BA) = \mathrm{tr}(I_n)$ ⇒ $\mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(I_n)$

=> $0 = n$ =><= (Contradiction)

Therefore there are no A, B that satisfies $AB - BA = I_n$. ■

4. Define a transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(a,b,c) = \begin{bmatrix} a+b \\ 2c-a \end{bmatrix}$$

Determine whether if T is a linear transformation or not.

Proof

Let $\mathbf{x}_1 = (a_1, b_1, c_1)$, $\mathbf{x}_2 = (a_2, b_2, c_2)$, and $k \in \mathbb{R}$.

$T(\mathbf{x}_1 + \mathbf{x}_2) = T(a_1+a_2, b_1+b_2, c_1+c_2)$

$$= \begin{bmatrix} (a_1+a_2)+(b_1+b_2) \\ 2(c_1+c_2)-(a_1+a_2) \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ 2c_1-a_1 \end{bmatrix} + \begin{bmatrix} a_2+b_2 \\ 2c_2-a_2 \end{bmatrix} = T(\mathbf{x}_1) + T(\mathbf{x}_2)$$

$T(k\mathbf{x}_1) = T(ka_1, kb_1, kc_1)$

$$= \begin{bmatrix} ka_1+kb_1 \\ 2kc_1-ka_1 \end{bmatrix} = k\begin{bmatrix} a_1+b_1 \\ 2c_1-a_1 \end{bmatrix} = kT(\mathbf{x}_1)$$

∴ T is a L.T. ■

Sage : http://math3.skku.ac.kr/home/pub/343

5. Give your **sketch** of the proof on Theorem 6.3.3:

"The Image of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$."

Sol

Show 1) $\mathrm{Im}(T)$ is closed under the vector addition.

2) $\mathrm{Im}(T)$ is closed under the scalar multiplication.

$\forall\, \mathbf{w}_1, \mathbf{w}_2 \in \mathrm{Im}\,T$, $\exists\, \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ such that $(\ni)$ $T(\mathbf{v}_1) = \mathbf{w}_1$, $T(\mathbf{v}_2) = \mathbf{w}_2$

⇒ $\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2)$

⇒ $\exists\, \mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{R}^n \ni T(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2 \in \mathbb{R}^m$ ∴ $\mathbf{w}_1 + \mathbf{w}_2 \in \mathrm{Im}\,T$

$\forall\, k \in \mathbb{R}$, $k\mathbf{w}_1 = kT(\mathbf{v}_1) = T(k\mathbf{v}_1)$

⇒ $\exists\, k\mathbf{v}_1 \in \mathbb{R}^n \ni T(k\mathbf{v}_1) = k\mathbf{w}_1 \in \mathbb{R}^m$ ∴ $k\mathbf{w}_1 \in \mathrm{Im}\,T$

∴ $\mathrm{Im}(T)$ is a subspace of $\mathbb{R}^m$. ■

VI. LA 2016-S, Participation and more (16pt) : Name: ________________

<Fill this form, Print it, Bring it and submit it just before your Midterm Exam on AM (9:00, April. 19th)

(시험전에 프린트하여, 빈칸을 채워서 제출하거나, 시험 중에 확인하여 채워서 시험 시간 중에 제출하면 됩니다.)

1. (10pt) Participations

(1) QnA Participations Numbers <Check yourself> : each weekly (From Sat – next Friday)

Week 1 : 1 2: 5 3: 6 4: 7

Week 5 : 8 6: 9 7; 10 8: 1

Total# : (Q: 20 A: 20)

Online Participation : 50 / 55 (1-8th week)

Off-line Participation/ Absence : 12 / 13 (2*7 - 1 holyday = 13 off line classes)

(2) Your Special Contribution :

: The number of your participations in Q&A with 'Finalized OK by SGLee' (No.),

(3) What are things that you have learned and recall well from QnA and PBL participation?

2. (5pt) Your presentation of Solutions in one Chapter.

(1) Your Team Number ( ) and Team members name ( )

(2) What was your chapter?

(3) What was your role in that process?

(4) How you can improve your presentation for Final and Final PBL Presentation?

3. (1pt, Bonus) Write anything you like to tell me.