Fall 2015, LA Midterm Exam Sol. (1 hour In class Exam)						Sign
Course	Linear Algebra	GEDB		Prof.		
Major		Year 학년		Student No. 학번		Name

※ Notice

1. Fillout the above boxes before you start this Exam. (학번, 이름 등을 기입하고 감독자 날인)
2. Honor Code: (시험 부정행위시 해당 교과목 성적이 "F" 처리됨은 물론 징계위원회에 회부될 수 있습니다.)
3. You can go out only after the permission from proctors.
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4. You may use the following <Sage codes> in your answers. (중간고사까지는 한국어 답안도 OK)

Total Score (100 pt)	
Offline Exam 85	Participation 15

```
var('a, b, c, d')                          # Define variables
eq1=3*a+3*b==12                            # Define equation1
eq2=5*a+2*b==13                            # Define equation2
solve([eq1, eq2], a,b)                     # Solve eq's
A=matrix(QQ, 3, 3, [3, 0, 0, 0, 0, 2, 0, 3, 4]);   # Matrix
x=vector([3, 1, 2])                        # Define vector x
A.augment(x)                               # [A: x]
A.echelon_form()                           # Find RREF
A.inverse()                                # Find inverse
A.det()                                    # Find determinant
A.adjoint()                                # Find adjoint matrix
A.charpoly()                               # Find charct. ploy
A.eigenvalues()                            # Find eigenvalues
A.eigenvectors_right()                     # Find eigenvectors
A.rank()                                   # Find rank of A
A.right_nullity()                          # Find nullity of A

A=random_matrix(QQ,7,7)   # random matrix of size 7 over Q
bool( A== B)              # Are A and B same?
P,L,U=A.LU()              # LU (P: Permutation M. / L, U

var('x, y')                                # Define variables
f = 7*x^2 + 4*x*y + 4*y^2-23               # Define a function
implicit_plot( f, (x, -10, 10), (y, -10, 10))   # implicit Plot
plot3d(y^2+1-x^3-x, (x, -pi, pi), (y, -pi, pi))  # 3D Plot

var('t')                                   # Define variables
x=2+2*t                                    # Define a parametric eq.
y=-3*t-2
parametric_plot((x,y), (t, -10, 10), rgbcolor='red')   # Plot
[G,mu]=A.gram_schmidt()     # G-S
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B #
A.jordan_form()     # Jordan Canonical Form of A
```

<Sample Sage Linear Algebra codes>

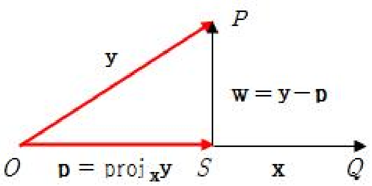
I. (1pt x 20= 20pt) True(T) or False(F).

1. (T) For each $\mathbf{y}$ and each subspace W of $\mathbb{R}^n$, the vector $\mathbf{y}-\text{proj}_W\mathbf{y}$ is orthogonal to W.
2. (F) A system of six linear equations with 3 unknowns cannot have more than 1 solution.
3. (T) A linear system of the form $A\mathbf{x}=\mathbf{0}$ containing eight equations and ten unknowns has infinitely many solutions.
4. (T) Not every linear independent set in $\mathbb{R}^n$ is an orthogonal set.
5. (T) Every linear system of the form $A\mathbf{x}=\mathbf{0}$ has at least 1 solution.
6. (T) A given matrix can be written uniquely as a sum of a symmetric matrix and a skew-symmetric matrix.
7. (F) Any subspace of $\mathbb{R}^2$ is either a line through the origin or $\mathbb{R}^2$.
8. (T) $\{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1-2x_3=0\}$ is a subspace of $\mathbb{R}^3$
9. (T) For any $n\times n$ matrix A with $n>1$, $\det(\text{adj}\,A)=\det(A)^{n-1}$.
10. (T) Let A be an $n\times n$ invertible matrix, then the inverse matrix of A is $A^{-1}=\frac{1}{|A|}\text{adj}\,A$.
11. (T) For a set of natural numbers $S=\{1,2,\ldots,n\}$, permutation is a one to one function from S to S.
12. (T) The determinant of matrix $A=[a_{ij}]$ in M_n, is defined as $\det(A)=\sum_{\sigma\in S_n}\text{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$.
13. (T) For any two $n\times n$ matrices A and B, $\det(AB)=\det(B)\det(A)$
14. (T) A matrix with all orthonormal columns is an orthogonal matrix.
15. (T) If the columns of an $m\times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x}\mapsto A\mathbf{x}$ preserves length.
16. (T) For any invertible lower triangular matrix A, A^{-1} is a lower triangular matrix.
17. (F) There is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ whose image is $\mathbb{R}^3$.
18. (F) For a transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, if $T(\mathbf{u})=T(\mathbf{v}) \Rightarrow \mathbf{u}=\mathbf{v}$, then it is called onto.
19. (F) For a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, $\text{Im}\,T$ is a subspace of $\mathbb{R}^n$.
20. (T) If a LT $T:\mathbb{R}^n\to\mathbb{R}^m$ is one-to-one and onto, then $n=m$ and T is called an isomorphism.

II. (2pt x 5 = 10pt) State or Define (Choose 5: Mark only 5 and Fill the boxes and/or state).

1. [$\text{proj}_{\mathbf{x}}\mathbf{y}$] The (vector) projection of $\mathbf{y}$ onto $\mathbf{x}$ and is denoted by $\text{proj}_{\mathbf{x}}\mathbf{y}$.

Here, the vector $\mathbf{w} = \overrightarrow{SP} = \mathbf{y} - \mathbf{p}$ is called the component of $\mathbf{y}$ orthogonal to $\mathbf{x}$. Therefore, $\mathbf{y}$ can be written as $\mathbf{y} = \mathbf{p} + \mathbf{w}$. For vectors $\mathbf{x}(\neq 0)$, $\mathbf{y}$ in $\mathbb{R}^3$, we have the following:



$$\text{proj}_{\mathbf{x}}\mathbf{y} = t\,\mathbf{x} \quad \text{where} \quad t = \frac{\mathbf{y}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}}.$$

2. [cofactor expansion] Let A be an $n \times n$ matrix. For any i, j $(1 \le i, j \le n)$ the following holds.

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \qquad \text{(cofactor expansion along the } i\text{th row)}$$

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \qquad \text{(cofactor expansion along the } j\text{th column)}$$

3. [eigenspace] Let A be an $n \times n$ matrix. For a nonzero vector $\mathbf{x} \in \mathbb{R}^n$, if there exist a scalar λ which satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then λ is called an eigenvalue of A, and $\mathbf{x}$ is called an eigenvector of A corresponding to λ.

Define an eigenspace of A corresponding to λ =

the solution space of the system of linear equations = $\{\mathbf{x} \in \mathbb{R}^n \mid (\lambda I_n - A)\mathbf{x} = 0\}$.

4. [kernel] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then

$$\ker T = \{\mathbf{v} \in \mathbb{R}^n \mid T(\mathbf{v}) = 0 \in \mathbb{R}^m\}$$

•

※ State the following concepts :

5. [Span of S]

the span of S is defined as the set of all linear combinations of elements of S.

6. [Linearly independent, linearly dependent]

$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly independent : $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0$

Otherwise, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are linearly dependent.

7. [Cramer's Rule]

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

let A be a coefficient matrix, and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$. Then the system of linear equations can be written as $A\mathbf{x} = \mathbf{b}$. If $|A| \neq 0$, the system of linear equations has a unique solution as follows:

$$x_1 = \frac{|A_1|}{|A|},\; x_2 = \frac{|A_2|}{|A|},\; \ldots,\; x_n = \frac{|A_n|}{|A|}$$

where A_j $(j = 1, 2, \cdots, n)$ denotes the matrix A with the jth column replaced by the vector $\mathbf{b}$.

III. (4pt x 7 = 28pts) Find or Explain (Fill the boxes) :

1. Find the distance D from the point $P(3,-1,2)$ to the plane $x+3y-2z-6=0$.

Sol $\quad \mathbf{p}=\mathrm{proj}_{\mathbf{n}}\mathbf{v}=t\mathbf{n}=\dfrac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}$.

Here, $\mathbf{n}=(1,3,-2)$, $\mathbf{v}=\overrightarrow{OP_0}-\overrightarrow{OP_1}=\mathbf{x}-\mathbf{x}_1=(3,-1,2)-(x_1,y_1,z_1)$ where $x_1+3y_1-2z_1-6=0$, so

$$\mathbf{p}=\mathrm{proj}_{\mathbf{n}}\mathbf{v}=\frac{(3-x_1,-1-y_1,2-z_1)\cdot(1,3,-2)}{1^2+3^2+(-2)^2}(1,3,-2)$$

$$=\frac{-x_1-3y_1+2z_1-4}{14}(1,3,-2)=\frac{-6-4}{14}(1,3,-2)$$

$$=-\frac{5}{7}(1,3,-2)=\left(-\frac{5}{7},-\frac{15}{7},\frac{10}{7}\right).$$

$$D=\|\mathrm{proj}_{\mathbf{n}}\mathbf{v}\|=\sqrt{\left(-\frac{5}{7}\right)^2+\left(-\frac{15}{7}\right)^2+\left(\frac{10}{7}\right)^2}=\frac{5\sqrt{14}}{7}$$ □

Sage Copy the following code into http://sage.skku.edu to practice.

```
n=vector([1, 3, -2])
v=vector([3, -1, 2]);d=-6
vn=v.inner_product(n)
nn=n.norm()
Distance=abs(vn+d)/nn
print Distance
```

5/7*sqrt(14) $\quad$ # $\dfrac{10}{\sqrt{14}}=\dfrac{5}{7}\sqrt{14}$ ■

2. Suppose that three points $(-1,7)$, $(2,15)$, $(1,3)$ pass through the parabola $y=a_0+a_1x+a_2x^2$. By plugging in these points, obtain three linear equations. Find coefficients a_0, a_1, a_2 by solving $A\mathbf{x}=\mathbf{b}$.

Sol

$$\begin{cases}a_0-a_1+a_2=7\\ a_0+2a_1+4a_2=15\\ a_0+a_1+a_2=3\end{cases}\quad(\because (-1,7),\ (2,15),\ (1,3) \text{ pass through the parabola})\quad \begin{bmatrix}1&-1&1\\1&2&4\\1&1&1\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}=\begin{bmatrix}7\\15\\3\end{bmatrix},\ \text{where } A=\begin{bmatrix}1&-1&1\\1&2&4\\1&1&1\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}7\\15\\3\end{bmatrix}.$$

$$[A:\mathbf{b}]=\left[\begin{array}{ccc:c}1&-1&1&7\\1&2&4&15\\1&1&1&3\end{array}\right]\xrightarrow{R_3-R_1}\left[\begin{array}{ccc:c}1&-1&1&7\\1&2&4&15\\0&2&0&-4\end{array}\right]\xrightarrow{\frac{1}{2}R_3}\left[\begin{array}{ccc:c}1&-1&1&7\\1&2&4&15\\0&1&0&-2\end{array}\right]\xrightarrow{R_2\leftrightarrow R_3}\cdots\rightarrow\left[\begin{array}{ccc:c}1&-1&1&7\\0&1&0&-2\\0&0&1&\frac{14}{3}\end{array}\right]\xrightarrow[R_2+R_1\to R_1]{-R_3+R_1\to R_1}\left[\begin{array}{ccc:c}1&0&0&\frac{1}{3}\\0&1&0&-2\\0&0&1&\frac{14}{3}\end{array}\right].$$

=> $a_0=\dfrac{1}{3},\ a_1=-2,\ a_2=\dfrac{14}{3}$. $\qquad$ Answer : $y=\dfrac{1}{3}-2x+\dfrac{14}{3}x^2$ ■

3. Let T_1 and T_2 are defined as follows:

$$T_1(x_1,x_2,x_3)=(4x_1,-2x_1+x_2,-x_1-3x_2),\qquad T_2(x_1,x_2,x_3)=(x_1+2x_2,-x_3,4x_1-x_3).$$

(1) Find the standard matrix for each T_1 and T_2.

(2) Find the standard matrix for each $T_2\circ T_1$ and $T_1\circ T_2$.

Sol

(1) $T_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}4\\-2\\-1\end{bmatrix},\ T_1\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\-3\end{bmatrix},\ T_1\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix},\qquad \therefore [T_1]=\begin{bmatrix}4&0&0\\-2&1&0\\-1&-3&0\end{bmatrix}$

$T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\4\end{bmatrix},\ T_2\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\0\\0\end{bmatrix},\ T_2\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\-1\\-1\end{bmatrix}\qquad \therefore [T_2]=\begin{bmatrix}1&2&0\\0&0&-1\\4&0&-1\end{bmatrix}$

(2) $[T_2\circ T_1]=[T_2][T_1]=\begin{bmatrix}1&2&0\\0&0&-1\\4&0&-1\end{bmatrix}\begin{bmatrix}4&0&0\\-2&1&0\\-1&-3&0\end{bmatrix}=\begin{bmatrix}0&2&0\\1&3&0\\17&3&0\end{bmatrix},\ [T_1\circ T_2]=[T_1][T_2]=\begin{bmatrix}4&0&0\\-2&1&0\\-1&-3&0\end{bmatrix}\begin{bmatrix}1&2&0\\0&0&-1\\4&0&-1\end{bmatrix}=\begin{bmatrix}4&8&0\\-2&-4&-1\\-1&-2&3\end{bmatrix}$ ■

Sage) http://math3.skku.ac.kr/home/pub/29 (See the code and output in page 4)

```
x,y,z=var('x y z')
A(x,y,z)=(4*x,-2*x+y,-x-3*y)
a(x,y,z)=(x+2*y,-z,4*x-z)
T=linear_transformation(QQ^3, QQ^3,A)
t=linear_transformation(QQ^3, QQ^3,a)
C = T.matrix(side='right')
c = t.matrix(side='right')
print "[T1]="
print C
print "[T2]="
print c
print "[T2*T1]="
print c*C
print "[T1*T2]="
print C*c
```

```
[T1]=            [T2]=
[ 4  0  0]       [ 1  2  0]
[-2  1  0]       [ 0  0 -1]
[-1 -3  0]       [ 4  0 -1]

[T2*T1]=         [T1*T2]=
[ 0  2  0]       [ 4  8  0]
[ 1  3  0]       [-2 -4 -1]
[17  3  0]       [-1 -2  3]
```

4. Let $H_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ moves any $\mathbf{x} \in \mathbb{R}^2$ to a symmetric image to a line which passes through the origin and has angle $\theta = \frac{\pi}{4}$ between the line and the x-axis. Find $H_\theta(\mathbf{x})$ for $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

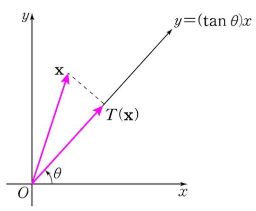
Sol The symmetric transformation H_θ which passes through the origin and has angle between the line and the x-axis is,

At $\theta = \frac{\pi}{4}$, $[H_\theta] = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\therefore\ H_\theta(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ ■

5. As shown in the picture, let us define an orthogonal projection as a linear transformation (linear operator) $P_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which maps any vector $\mathbf{x}$ in $\mathbb{R}^2$ to the orthogonal projection on a line, which passes through the origin with angle $\theta = \frac{\pi}{4}$ between the x-axis and the line. Let us denote the standard matrix corresponding to P_θ when $H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

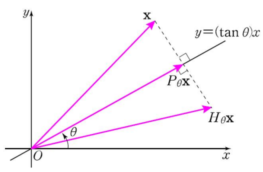
Sol $P_\theta \mathbf{x} - \mathbf{x} = \frac{1}{2}(H_\theta \mathbf{x} - \mathbf{x})$ (the same direction with a half length)



$P_\theta \mathbf{x} = \frac{1}{2} H_\theta \mathbf{x} + \frac{1}{2}\mathbf{x} = \frac{1}{2} H_\theta \mathbf{x} + \frac{1}{2} I\mathbf{x} = \frac{1}{2}(H_\theta + I)\mathbf{x}$

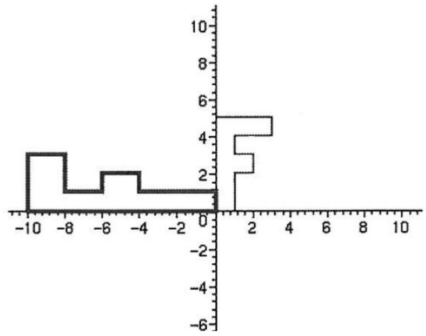
$P_\theta = \frac{1}{2}(H_\theta + I) = \left(\begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1-\cos 2\theta) \end{bmatrix} \right.$

$\Rightarrow \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}_{\theta=\frac{\pi}{4}} = \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right)$. ■



6. Find a linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger **F**.):

Sol



Answer : $T(A) = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$

since $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta = \frac{\pi}{2}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$. ■

7. [Invertible Matrix Theorem] Let A be an $n \times n$ matrix.

Which of the following statements is not equivalent to "the matrix A is invertible."? (Choose one)

(1) Column vectors of A are linearly independent.
(2) Row vectors of A are linearly independent.
(3) $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = \mathbf{0}$.
(4) For any $n \times 1$ vector $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.
(5) A and I_n are row equivalent.
(6) A and I_n are column equivalent.
(7) $\det(A) \neq 0$
(8) $\lambda = 0$ is an eigenvalue of A.
(9) $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_A(\mathbf{x}) = A\mathbf{x}$ is one-to-one.
(10) $T_A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T_A(\mathbf{x}) = A\mathbf{x}$ is onto.

Ans 8 . ■

IV. (3+4+5=12pt) Python/ Sage Computations.

1. (3pts) When we solve a LSE $A\mathbf{x} = \mathbf{b}$ whose augmented matrix is $B = \left[\begin{array}{cccc:c} 2 & 1 & 1 & -2 & 1 \\ 3 & -2 & 1 & -6 & -2 \\ 1 & 1 & -1 & -1 & -1 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$ and $\text{RREF}(B) = \left[\begin{array}{cccc:c} 1 & 0 & 0 & -\frac{17}{11} & 0 \\ 0 & 1 & 0 & \frac{9}{11} & 0 \\ 0 & 0 & 1 & \frac{3}{11} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

Explain why this system has no solution.

Ans The last equation in the system means $w\,0 = 1$ which is impossible when $\mathbf{x} = (x, y, z, w)$ is a solution. Therefore $A\mathbf{x} = \mathbf{b}$ has a solution set is $\varnothing$ (Empty set).■

2. **(4pts)** Consider $A\mathrm{x}=\mathrm{y}$ where $A=\begin{bmatrix}0&1&2&3\\1&2&3&4\\2&3&4&5\\3&4&5&6\end{bmatrix}$ and $\mathrm{y}=\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$. Similarly we have found the augmented matrix $[A:\mathrm{y}]$ and its

RREF by Sage RREF($[A:\mathrm{y}]$) $=\begin{bmatrix}1&0&-1&-2&2\\0&1&2&3&-1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$

(1) Find number of linear independent rows of A Ans: (2)

(2) The solution set of $A\mathrm{x}=\mathrm{y}$.

Ans: $\{(s+2t+2,\ -2s-3t-1,\ s,\ t)\,|\,s,t\in\mathbb{R}\}$ **or** $\left\{\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}+s\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}+t\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}\,\middle|\, s,t\in\mathbb{R}\right\}$ ■

3. **(5pts)** Consider $A\mathrm{x}=\mathrm{y}$ where $A=\begin{bmatrix}-18&-30&-30&-36\\42&54&30&36\\-6&-6&18&0\\30&30&30&48\end{bmatrix}$ and $\mathrm{y}=\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}$. You were asked to find

(1) Augment matrix $[A:\mathrm{y}]$ (2) RREF(A) (3) Det$A$ (4) Inverse of A (4) characteristic polynomial of A (5) all eigenvalues of A (6) all eigenvectors of A. The following is your answer. Fill out the blanks to find each.

Sol)

1) Step 1: Browse http://math3.skku.ac.kr or http://math1.skku.ac.kr/ (or http://sage.skku.edu/ or https://cloud.sagemath.com etc)
2) Step 2: Type class/your ID: (math2013 or yours) and PW : (math**** or yours)
3) Step 3: Click "New worksheet (새 워크시트)" button.
4) Step 4: Define a matrix A in the first cell in rational (QQ) field.

```
A = matrix(QQ, 4, 4, [-18, -30,-30,-36,42,54,30,36,-6,-6,18,0,30,30,30,48]) and y = matrix(QQ, 4, 1, [-1, 0, 1, 2])
```

5) Step 5: Type a command to find augment matrix **[A: y]** A.augment(y) and evaluate
6) Step 6: Type a command to find **RREF**(A) **A.echelon_form()** and evaluate.
7) Step 7: Type a command to find **determinant of** A **A.det()** and evaluate.
8) Step 8: Type a command to find inverse of A **A.inverse()** and evaluate.
9) Step 9: Type a command to find char. polynomial of A **A.charpoly()** and evaluate.
10) Step 10: Type a command to find **eigenvalues** of A **A.eigenvalues()** and evaluate.
11) Step 11: Type a command to find **eigenvectors** of A **A.eigenvectors_right()** and evaluate.
13) Last step : Give 'print' command to see what you like to read.

Now we have some out from the Sage.

RREF(A) = Identity matrix of size 4

det(A) = 248832

inverse(A) =

```
[ 17/144   5/144   5/144    1/16]
[-11/144   1/144  -5/144   -1/16]
[   1/72    1/72    1/18       0]
[ -5/144  -5/144  -5/144    1/48]
```

characteristic polynomial of (A) = x^4 - 102*x^3 + 3528*x^2 - 50112*x + 248832

eigenvalues of A = { 48, 24, 18, 12}

eigenvectors = [(48, [(1, -1, 0, -1)], 1), (24, [(0, 1, -1, 0)], 1), (18, [(1, -1, 1, -1)], 1), (12, [(1, -1, 0, 0)], 1)]

Write what (24, [(0, 1, -1, 0)], 1) means in eigenvectors of A :

24 : eigenvalue, [(0, 1, -1, 0)] : corresponding eigenvector , 1 : algebraic multiplicity of engenvalue 24 ,

■

V. (3pt x 5 = 15pt) Explain or give a sketch of proof.

1. If $A^2 = A$, show that $(I-2A) = (I-2A)^{-1}$.

Proof Show $(I-2A)(I-2A) = I$ when $A^2 = A$

$$(I-2A)(I-2A) = I - 2A - 2A + 4A^2$$
$$= I - 4A + 4A = I \quad (\because A^2 = A)$$

$$\therefore (I-2A)^{-1} = (I-2A) \quad \blacksquare$$

2. Show AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ when A, B are invertible square matrices of order n.

Proof $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= AI_nA^{-1} = AA^{-1} = I_n. \quad \blacksquare$$

3. Let A and I be $n \times n$ matrices. If $A+I$ is invertible, show that $A(A+I)^{-1} = (A+I)^{-1}A$.

Proof $(A+I)A = A^2 + A = A(A+I)$

=> $(A+I)^{-1}(A+I)A(A+I)^{-1} = (A+I)^{-1}A(A+I)(A+I)^{-1}$ ($\because$ $A+I$ is invertible)

=> $A(A+I)^{-1} = (A+I)^{-1}A$ $\blacksquare$

4. Show $W_6 = \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\}$ is a subspace of $\mathbb{R}^3$.

Sol

Show 1) W_6 is closed under the vector addition.

2) W_6 is closed under the scalar multiplication.

$\forall$ $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (x_4, x_5, x_6) \in W$, $k \in \mathbb{R}$

1) $\mathbf{x} + \mathbf{y} = (x_1 + x_4,\ x_2 + x_5,\ x_3 + x_6) \in W_6$ $\quad (\because x_1 + x_4 = x_2 + x_5 = x_3 + x_6)$

2) $k\mathbf{x} = (kx_1,\ kx_2,\ kx_3) \in W_6$ $\quad (\because kx_1 = kx_2 = kx_3)$

Therefore, W_6 is a subspace of $\mathbb{R}^3$. $\blacksquare$

5. Show the following :

Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be vector spaces and $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Then T is one-to-one **if and only if** $\ker T = \{0\}$.

Proof ($\Rightarrow$) As $\forall\, \mathrm{v} \in \ker T$, $T(\mathrm{v}) = 0 = T(0)$ and T is one-to-one,

$\Rightarrow$ $\mathrm{v} = 0$ $\qquad \therefore \ker T = \{0\}$

($\Leftarrow$) $T(\mathrm{v}_1) = T(\mathrm{v}_2)$ $\Rightarrow$ $0 = T(\mathrm{v}_1) - T(\mathrm{v}_2) = T(\mathrm{v}_1 - \mathrm{v}_2)$

$\Rightarrow$ $\mathrm{v}_1 - \mathrm{v}_2 \in \ker T = \{0\}$ $\Rightarrow$ $\mathrm{v}_1 = \mathrm{v}_2$

$\therefore$ T is one-to-one. $\blacksquare$

VI. Participation and more (15pt) :

Name: ____________________

<Fill this form, Print it, Bring it and submit it just before your Midterm Exam on AM 10:30, Oct. 20th)

1. (10pt) Participations

(1) QnA Participations Numbers <Check yourself> : each weekly (From Sat - next Friday)

Week 1 : 5 2: 5 3: 5 4: 5

Week 5 : 5 6: 5 7: 5 (8: 0)

Total# : (Q: A:)

Online Participation : 31 / 33

Off-line Participation/ Absence : 12 / 13

(2) Your Special Contribution : including The number of your participations in Q&A with Finalized OK by SGLee (No.), Your valuable comments on errata (No.) or shared valuable informations and others (No.)

(3) What are things that you have learned and recall well from the above participation?

2. (5pt) Project Proposal and/or Your Constructive suggestions

Title(Tentative), Goals and Objectives of your possible project:

** Linear Algebra in ??? Engneering ***

< Some of you made a good Project Proposal but not in general. Need to improve.>

SKKU LA 2015 PBL 보고서 발표 by 김** & 우**, http://youtu.be/hUDuQ8e8HsU

SKKU 선형대수학 PBL 보고서 발표 by 손** http://youtu.be/woyS_EYWiDs

SKKU 선형대수학 PBL 보고서 ppt 발표 by 박** http://youtu.be/E-5m65-8Ea8

Motivation and Significance of your possible project:

** My major and career ***

Working Plan:

** Team with ***

Web Resources (addresses) / References (book etc) : *****

선형대수학 자료실: http://matrix.skku.ac.kr/LinearAlgebra.htm

선형대수학 거꾸로 교실 자료 : http://matrix.skku.ac.kr/SKKU-LA-FL-Model/SKKU-LA-FL-Model.htm

* 선형대수학 강좌 운영방법 소개 동영상 : http://youtu.be/Mxp1e2Zzg-A

* 선형대수학 강좌 기록 일부 http://matrix.skku.ac.kr/2015-LA-FL/SKKU-LA-Model.pdf

http://matrix.skku.ac.kr/2015-LA-FL/Linear-Algebra-Flipped-Class-SKKU.htm

(Sample: http://www.prenhall.com/esm/app/ph-linear/kolman/html/proj_intro.html

http://home2.fvcc.edu/~dhicketh/LinearAlgebra/LinAlgStudentProjects.html

http://www.math.utah.edu/~gustafso/s2012/2270/projects.html

http://www2.stetson.edu/~mhale/linalg/projects.htm etc)

Etc: Write anything you like to tell me.

*****

Fall 2015, LA Final Comprehensive Exam (In class Exam: Solution)						Sign
Course	Linear Algebra	GEDB		Prof.		
Major		Year (학년)		Student No. (학번)		Name

※ Notice

1. Fill out the above boxes before you start this Exam. (학번, 이름 등을 기입하고 감독자 날인)
2. Honor Code: (시험 부정행위시 해당 교과목 성적이 "F" 처리됨은 물론 징계위원회에 회부될 수 있습니다.)
3. You can go out only after the permission from proctors.
(감독위원의 지시가 있기 전에는 고사장 밖으로 나갈 수 없으며, 감독위원의 퇴실 지시가 있으면 답안지를 감독위원께 제출한 후에 퇴실하시기 바랍니다.)
4. You may use the following <Sage codes> in your answers.
http://matrix.skku.ac.kr/LinearAlgebra.htm http://matrix.skku.ac.kr/LA-Lab/

Total Score = 100 pt

```
var('a, b, c, d')                        # Define variables
eq1=3*a+3*b==12                          # Define equation1
eq2=5*a+2*b==13                          # Define equation2
solve([eq1, eq2], a,b)                   # Solve eq's
A=matrix(QQ, 3, 3, [3, 0, 0, 0, 0, 2, 0, 3, 4]);   # Matrix
x=vector([3, 1, 2])                      # Define vector x
A.augment(x)                             # [A: x]
A.echelon_form()                         # Find RREF
A.inverse()                              # Find inverse
A.det()                                  # Find determinant
A.adjoint()                              # Find adjoint matrix
A.charpoly()                             # Find charct. poly
A.eigenvalues()                          # Find eigenvalues
A.eigenvectors_right()                   # Find eigenvectors
A.rank()                                 # Find rank of A
A.right_nullity()                        # Find nullity of A
var('t')                                 # Define variables
x=2+2*t                                  # Define a parametric eq.
y=-3*t-2
bool( A== B)                             # Are A and B same?
```

```
var('x, y')                              # Define variables
f = 7*x^2 + 4*x*y + 4*y^2-23             # Define a function
implicit_plot( f, (x, -10, 10), (y, -10, 10))   # implicit Plot
parametric_plot((x,y), (t, -10, 10), rgbcolor='red')   # Plot
plot3d(y^2+1-x^3-x, (x, -pi, pi), (y, -pi, pi))   # 3D Plot
A=random_matrix(QQ,7,7)     # random matrix of size 7 over Q
F=random_matrix(RDF,7,7)    # random matrix of size 7 over Q
P,L,U=A.LU()                # LU (P: Permutation M. / L, U
print P, L, U

h(x, y, z) = [x+2*y-z, y+z, x+y-2*z]
T = linear_transformation(U, U, h)   # L.T.
print T.kernel()                     # Find a basis for kernel(T)

C=column_matrix([x1, x2, x3])
D=column_matrix([y1, y2, y3])
aug=D.augment(C, subdivide=True)
Q=aug.rref()            # Find a matrix representation of T.

[G,mu]=A.gram_schmidt()     # G-S
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B #

A.H   # conjugate transpose of A
A.jordan_form()    # Jordan Canonical Form of A
```

<Sample Sage Linear Algebra codes>

I. (2pt x 15= 30pt) True(T) or False(F).

1. (F) An eigenvalue of A is the same as an eigenvalue of the reduced row echelon form of A. *(not same)*
2. (T) For any $n\times n$ singular matrix A with $n>1$, $\det(\text{adj}\,A)=\det(A)^{n-1}$.
3. (T) For $\mathbf{y}\neq 0$ and a non-trivial subspace W (of $\mathbb{R}^n$) not containing $\mathbf{y}$, the vector $\mathbf{y}-\text{proj}_W\mathbf{y}$ is orthogonal to W.
4. (F) If one replaces a matrix with its transpose, then the image, rank, and kernel may change, but the nullity does not change.
5. (T) If $S^{-1}AS\mathbf{x}=\lambda\mathbf{x}$ and $\mathbf{x}$ is a non-zero vector, then $S\mathbf{x}$ is an eigenvector of A corresponding to λ.

6. (T) If $A\in M_n$ is diagonalizable, then A should have n linearly independent eigenvectors.
7. (F) Let $A=[A^{(1)}\ A^{(2)}\ \cdots\ A^{(6)}]$ be a 4×6 matrix. If the columns of A spans a 4-dimensional subspace of $\mathbb{R}^4$, then the set $\{A^{(1)}, A^{(2)}, \cdots, A^{(6)}\}$ is linearly independent. *(L. D. : 6 vectors in $\mathbb{R}^4$)*
8. (F) If $\mathbf{u}_1,\cdots,\mathbf{u}_r,\mathbf{v}_1,\cdots,\mathbf{v}_s$ are linearly independent vectors, then $<\mathbf{u}_1,\cdots,\mathbf{u}_r>\cap<\mathbf{v}_1,\cdots,\mathbf{v}_s>\neq\{0\}$. $(=\{0\})$
9. (F) The set $\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix}\,\middle|\, a-d=0\right\}$ is a subspace of $(M_2,+,\cdot)$. *(not a subspace)*
10. (F) The vectors $\mathbf{x}_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}, \mathbf{x}_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}, \mathbf{x}_3=\begin{bmatrix}1&1\\1&0\end{bmatrix}, \mathbf{x}_4=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ form a basis for $(M_2,+,\cdot)$. *(L.D. : $\mathbf{x}_1+\mathbf{x}_4=\mathbf{x}_3$)*

11. (F) Let A be a square matrix. If $A^*=-A \Rightarrow \text{tr}(A)=0$. *(It is true when $A^*=A$)*
12. (T) Let A be a positive definite and symmetric matrix of order 3. Then $<\mathbf{x},\mathbf{y}>\ =\mathbf{x}^TA\,\mathbf{y}$ is an inner product on $\mathbb{R}^3$.
13. (T) If A is an $n\times n$ diagonalizable matrix with a diagonalizing matrix X, then the matrix $Y=(X^{-1})^T$ diagonalize A^T.
14. (F) Every eigenvalue of a Hermitian matrix is purely imaginary. *(all real eigenvalues)*
15. (T) If A is a square matrix, then A^*A is unitary diagonalizable.

II. (5*2pt = 10pt) Define and/or State :
※ Choose 3 of the following.

Kernel, linearly independent, Cofactor expansion, Real orthogonal Matrix, one-to-one, onto, Vector space (V, ⊕, ∘), Hermitian matrix, Unitary matrix, Normal matrix, Inner product space

1. Definition [Kernel]

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The set of all vectors in $\mathbb{R}^n$, whose image becomes $\mathbf{0}$ by T, is called **kernel** of T and is denoted by $\ker T$. That is, $\ker T = \{\mathbf{v} \in \mathbb{R}^n \mid T(\mathbf{v}) = \mathbf{0}\}$.

Definition [onto]

For a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, if there exist $\mathbf{v} \in \mathbb{R}^n$ for any given $\mathbf{w} \in \mathbb{R}^m$, such that $T(\mathbf{v}) = \mathbf{w}$, then it is called **onto** (surjective).

[Cofactor expansion]

Let A be an $n \times n$ matrix. For any i, j $(1 \le i, j \le n)$ the following holds.

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ (cofactor expansion along the ith row)

$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ (cofactor expansion along the jth column)

2. Definition [Vector space]

If a set $V(\neq \phi)$ has two well-defined binary operations, vector addition (A) '+' and scalar multiplication (SM) '$\cdot$', and for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $h, k \in \mathbb{R}$, they are closed under vector addition and scalar multiplication (i.e. satisfies two basic laws) and also satisfies 8 operational laws, then we say that the set V forms a **vector space** over $\mathbb{R}$ with the given two operations, and we denote it by $(V, +, \cdot)$ (simply V if there is no confusion). Elements of V are called **vectors**.

3. [Normal matrix]

If matrix $A \in M_n(\mathbb{C})$ satisfies $AA^* = A^*A$, then A is called a **normal matrix**.

Definition [Inner product and inner product space]

The **inner product** on a real vector space V is a function assigning a pair of vectors $\mathbf{u}$, $\mathbf{v}$ to a scalar $<\mathbf{u}, \mathbf{v}>$ satisfying the following conditions. (that is, the function $< \ , \ > : V \times V \to \mathbb{R}$ satisfies the following conditions.)

(1) $<\mathbf{u}, \mathbf{v}> = <\mathbf{v}, \mathbf{u}>$ for every $\mathbf{u}, \mathbf{v}$ in V.

(2) $<\mathbf{u}+\mathbf{v}, \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$ for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V.

(3) $<c\mathbf{u}, \mathbf{v}> = c<\mathbf{u}, \mathbf{v}>$ for every $\mathbf{u}, \mathbf{v}$ in V and c in $\mathbb{R}$.

(4) $<\mathbf{u}, \mathbf{u}> \ge 0$; $<\mathbf{u}, \mathbf{u}> = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$ for every $\mathbf{u}$ in V.

The **inner product space** is a vector space V with an inner product $<\mathbf{u}, \mathbf{v}>$ defined on V.

※ Choose 2 of the following.

Singular values of matrix A, SVD, LDU and QR decomposition, Gram-Schmidt Orthonomalization process,
Wronski's Test, Quadratic form $\mathrm{x}^T A \mathrm{x}$, Least square solution, Pseudo-inverse, Schur's Theorem,
Jordan block, Generalized Eigenvector

4. [Singular values of matrix A, the singular value decomposition (SVD) of A]

Let A be an $m \times n$ real matrix. Then there exist orthogonal matrices U of order m and V of order n, and an $m \times n$ matrix Σ such that $U^T A V = \begin{pmatrix} \Sigma_1 & O \\ O & O \end{pmatrix} = \Sigma$, (1)

where the main diagonal entries of Σ_1 are positive and listed in the monotonically decreasing order, and O is a zero-matrix. That is,

$$A = U\Sigma V^T = [\mathrm{u}_1 \mathrm{u}_2 \cdots \mathrm{u}_k \mathrm{u}_{k+1} \cdots \mathrm{u}_m] \left[\begin{array}{cccc|ccc} \sigma_1 & & & 0 & 0 & \cdots & 0 \\ & \sigma_2 & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & & \vdots \\ 0 & & & \sigma_k & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] \begin{bmatrix} \mathrm{v}_1^T \\ \mathrm{v}_2^T \\ \vdots \\ \mathrm{v}_n^T \end{bmatrix}, \text{ where } \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0.$$

Equation (1) is called the singular value decomposition (SVD) of A. The main diagonal entries of the matrix Σ are called the singular values of A. In addition, the columns of U are called the left singular vectors of A and the columns of V are called the right singular vectors of A.

Definition **[Pseudo-Inverse]**

For an $m \times n$ matrix A, the $n \times m$ matrix $A^{\dagger} = V\Sigma' U^T$ is called a pseudo-inverse of A, where U, V are orthogonal matrices in the SVD of A and Σ' is

$$\Sigma' = \begin{bmatrix} \Sigma_1^{-1} & O \\ O & O \end{bmatrix} \text{ (where } \Sigma_1 \text{ is nonsingular).}$$

5. [Schur's Theorem]

A square matrix A is unitarily similar to an upper triangular matrix whose main diagonal entries are the eigenvalues of A. That is, there exists a unitary matrix U and an upper triangular matrix T such that

$$U^* A U = T = [t_{ij}] \in M_n(\mathbb{C}),\ t_{ij} = 0 (i > j),$$

where t_{ii}'s are eigenvalues of A.

[Jordan canonical form] (The Jordan Canonical Form (JCF) of a matrix A is a block diagonal matrix composed of Jordan blocks, each with eigenvalues of A on its respective diagonal, 1's on its super-diagonal, and 0's elsewhere.) Let A be an $n \times n$ matrix with t ($1 \leq t \leq n$) linearly independent eigenvectors. Then,

A is similar to a matrix $J_A = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ & & \ddots \\ 0 & & J_t \end{bmatrix}_{n \times n}$

where $U^* A U = J_A$ for some unitary matrix U. Furthermore, we have

$$J_k = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}_{n_k \times n_k}, \quad (n_1 + n_2 + \cdots + n_t = n,\ 1 \leq k \leq t)$$

where each J_k, called a Jordan block, corresponds to an eigenvalue λ_i of A. The block diagonal matrix J_A is called the Jordan canonical form of A and each J_k are called Jordan blocks of J_A.

III. (4pt x 7 = 28pts) Find : Fill the boxes

1. Find the distance D from the point $P(3,-1,2)$ to the plane $x+3y-2z-6=0$.

Sol Here, $\mathbf{n}=(1,3,-2)$, $\mathbf{v}=\overrightarrow{OP_0}-\overrightarrow{OP_1}=\mathbf{x}-\mathbf{x}_1=(3,-1,2)-(x_1,y_1,z_1)$ where $x_1+3y_1-2z_1-6=0$, so

$$\mathbf{p}=\mathrm{proj}_{\mathbf{n}}\mathbf{v}=\frac{\boxed{\mathbf{v}}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}=\frac{(3-x_1,-1-y_1,2-z_1)\cdot(1,3,-2)}{1^2+3^2+(-2)^2}(1,3,-2)$$

$$=\frac{-x_1-3y_1+2z_1-4}{14}(1,3,-2)=\frac{-6-4}{14}(1,3,-2)$$

$$=-\frac{5}{7}(1,3,-2)=(-\frac{5}{7},-\frac{15}{7},\frac{10}{7}).$$

$$D=\|\mathrm{proj}_{\mathbf{n}}\mathbf{v}\|=\boxed{\sqrt{(-\frac{5}{7})^2+(-\frac{15}{7})^2+(\frac{10}{7})^2}}=\boxed{\frac{5\sqrt{14}}{7}}.$$ ■

2. Let $A=\begin{bmatrix}1&2&1&3&2\\3&4&9&0&7\\2&3&5&1&8\\2&2&8&-3&5\end{bmatrix}$. Then RREF of A^T is given as follows:

$$\begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Then a basis for the column space of A is $\boxed{\{(1,0,0,-1),(0,1,0,1),(0,0,1,0)\}}$ and nullity of A is $\boxed{2}$. ■

3. Find coefficients a,b,c of the parabolic equation $y=a+bx+cx^2$ which passes through the three points $(1,3)$, $(2,3)$, and $(3,5)$.

Sol Let $y=a+bx+cx^2$ $\quad 3=a+b(1)+c(1)^2$, $3=a+b(2)+c(2)^2$, $5=a+b(3)+c(3)^2$ $\quad\therefore$ Vandermonde matrix is $V=\begin{pmatrix}1&1&1\\1&2&4\\1&3&9\end{pmatrix}$. $V\begin{bmatrix}a\\b\\c\end{bmatrix}=V\mathbf{x}=\mathbf{b}=\begin{bmatrix}3\\3\\5\end{bmatrix}$

$$\det V=\begin{vmatrix}1&1&1\\1&2&4\\1&3&9\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&3\\0&2&8\end{vmatrix}=8-6=2,\quad V^{-1}=\frac{1}{|V|}\mathrm{adj}\,V=\frac{1}{2}\begin{bmatrix}6&-6&2\\-5&8&-3\\1&-2&1\end{bmatrix}=\begin{bmatrix}3&-3&1\\-\frac{5}{2}&4&-\frac{3}{2}\\\frac{1}{2}&-1&\frac{1}{2}\end{bmatrix}$$

$$\therefore\begin{bmatrix}a\\b\\c\end{bmatrix}=V^{-1}\begin{bmatrix}3\\3\\5\end{bmatrix}=\boxed{\begin{bmatrix}3&-3&1\\-\frac{5}{2}&4&-\frac{3}{2}\\\frac{1}{2}&-1&\frac{1}{2}\end{bmatrix}\begin{bmatrix}3\\3\\5\end{bmatrix}=\begin{bmatrix}5\\-3\\1\end{bmatrix}}.$$

Ans: $y=\boxed{5-3x+x^2}$ http://math1.skku.ac.kr/home/pub/2475 ■

4. (1) Show that $S=\{\mathbf{v}_1=(0,0,1,0),\mathbf{v}_2=(1,0,1,1),\mathbf{v}_3=(1,1,2,1)\}\subset\mathbb{R}^4$ is linearly independent, and
(2) find its corresponding orthonormal set.

Sol (1) Let $c_1,c_2,c_3\in\mathbb{R}$.

$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}=(0,0,0,0)$

$\Rightarrow(c_2+c_3,\ c_3,\ c_1+c_2+2c_3,\ c_2+c_3)=\mathbf{0}$

$$\Rightarrow\begin{cases}c_2+c_3=0\\c_1+c_2+2c_3=0\\c_3=0\end{cases}$$

Since $\boxed{c_1=c_2=c_3=0}$, S is linearly independent. ■

(2) Using Gram-Schmidt orthonormal process,

$\Rightarrow\mathbf{y}_1=\mathbf{v}_1=(0,0,1,0)$

$$\Rightarrow\mathbf{y}_2=\mathbf{v}_2-\mathrm{proj}_{W_1}\mathbf{v}_2=\mathbf{v}_2-\frac{\mathbf{v}_2\cdot\mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1=(1,0,1,1)-(0,0,1,0)=(1,0,0,1)$$

$$\Rightarrow\mathbf{y}_3=\mathbf{v}_3-\mathrm{proj}_{W_2}\mathbf{v}_3=\mathbf{v}_3-\frac{\mathbf{v}_3\cdot\mathbf{y}_1}{\|\mathbf{y}_1\|^2}\mathbf{y}_1-\frac{\mathbf{v}_3\cdot\mathbf{y}_2}{\|\mathbf{y}_2\|^2}\mathbf{y}_2=(1,1,2,1)-(0,0,2,0)-(1,0,0,1)=(0,1,0,0)$$

Hence an orthonormal set corresponding to S, $Z=\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\}$ is $\boxed{Z=\left\{(0,0,1,0),(\frac{\sqrt{2}}{2},0,0,\frac{\sqrt{2}}{2}),(0,1,0,0)\right\}}$ ■

5. For u_1=(1,2), u_2=(2,3), v_1=(1,3), v_2=(1,4), let α={u_1,u_2}, β={v_1,v_2} be two ordered bases for $\mathbb{R}^2$. Find the transition matrix $[I]_\beta^\alpha$ and $[I]_\alpha^\beta$.

Sol (1) $P = [I]_\beta^\alpha = [\,[v_1]_\alpha : [v_2]_\alpha\,]$

$\Rightarrow [u_1 : u_2 : v_1 : v_2] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 4 \end{bmatrix}$

$\Rightarrow$ RREF form : $\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -1 & -2 \end{bmatrix}$

$\therefore\ P = [I]_\beta^\alpha = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$ and $[I]_\alpha^\beta = P^{-1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$. ■

6. Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$. Find a matrix P which is an orthogonally diagonalizing matrix A.

Sol (By Sage: A=matrix(3, 3, [2, 2, 2, 2, 1, -1, 2, -1, 1]), print A.eigenvectors_right(), (4, [(2, 1, 1)], 1), (2, [(0, 1, -1)], 1), (-2, [(1, -1, -1)], 1)])

Since $P_A(\lambda) = |\lambda I - A| = (\lambda+2)(\lambda-2)(\lambda-4) = 0$, we obtain eigenvalues of A ($\lambda_1 = 4, \lambda_2 = 2$ and $\lambda_3 = -2$).

The vector $(2,1,1)$ is an eigenvector corresponding to $\lambda_1 = 4$, $(0,1,-1)$ is an eigenvector corresponding to $\lambda_2 = 2$ and $(1,-1,-1)$ is an eigenvector corresponding to $\lambda_3 = -2$. Hence **the unitary (real orthogonal) matrix** P is given by

$$P = \begin{bmatrix} \frac{2}{\sqrt6} & 0 & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt2} & -\frac{1}{\sqrt3} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{\sqrt6}{3} & 0 & \frac{\sqrt3}{3} \\ \frac{\sqrt6}{6} & \frac{\sqrt2}{2} & -\frac{\sqrt3}{3} \\ \frac{\sqrt6}{6} & -\frac{\sqrt2}{2} & -\frac{\sqrt3}{3} \end{bmatrix}$$

(signs in col's can be changed)

such that $P^TAP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. ■

7. Using SVD(Singular Value Decomposition), find a pseudo-inverse of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, that has the full column rank.

Sol [We could also find it with $A^\dagger = (A^TA)^{-1}A^T$ **when** A **has a full column rank.**]

(But in general) We first compute the SVD of A (You may use http://matrix.skku.ac.kr/2014-Album/MC.html)

$$A = [u_1\ u_2]\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} = \begin{bmatrix} \frac{\sqrt6}{3} & 0 \\ \frac{\sqrt6}{6} & -\frac{\sqrt2}{2} \\ \frac{\sqrt6}{6} & \frac{\sqrt2}{2} \end{bmatrix}\begin{bmatrix} \sqrt3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt2}{2} & \frac{\sqrt2}{2} \\ \frac{\sqrt2}{2} & -\frac{\sqrt2}{2} \end{bmatrix}.$$

Then $A^\dagger = [v_1\ v_2]\begin{bmatrix} \frac{1}{\sigma_1} & 0 \\ 0 & \frac{1}{\sigma_2} \end{bmatrix}\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} =$

$$= \begin{bmatrix} \frac{\sqrt2}{2} & \frac{\sqrt2}{2} \\ \frac{\sqrt2}{2} & -\frac{\sqrt2}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt6}{3} & \frac{\sqrt6}{6} & \frac{\sqrt6}{6} \\ 0 & -\frac{\sqrt2}{2} & \frac{\sqrt2}{2} \end{bmatrix}$$

$$\text{or } \begin{bmatrix} \frac{\sqrt2}{2} & \frac{\sqrt2}{2} \\ \frac{\sqrt2}{2} & -\frac{\sqrt2}{2} \end{bmatrix}\begin{bmatrix} \frac{\sqrt3}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt6}{3} & \frac{\sqrt6}{6} & \frac{\sqrt6}{6} \\ 0 & -\frac{\sqrt2}{2} & \frac{\sqrt2}{2} \end{bmatrix} = \begin{bmatrix} \frac13 & -\frac13 & \frac23 \\ \frac13 & \frac23 & -\frac13 \end{bmatrix}$$ ■

IV. (12pts) Solve them with Python/Sage Codes.

1. You did define a matrix $A=\begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix}$ and other works as following in a Sage Cell http://sage.skku.edu/.

And you found the following output. Please explain what you did in empty spaces as much as you can.

```
# Sage Cell http://sage.skku.edu/.
A=matrix(4, 4, [2, -4, 2, 2, -2, 0, 1, 3, -2, -2, 3, 3, -2, -6, 3, 7])
#A = random_matrix(QQ, 7, 7) What this means? Generate the random 7×7 matrix in Q, # mean Do not evaaluate
y = matrix(QQ, 4, 1, [-1, 0, 1, 2])
F= A.augment(y)
print F
print F.echelon_form()
print A.det()
print A.inverse()
print A.charpoly()
print A.eigenvalues()
print A.eigenvectors_right()                What this means?
    Find (right) eigenvalues (and eigenvectors of A including algebraic and geometric multiplicities of each eigenvalues.)
U = QQ^3
x, y, z = var('x, y, z')
h(x, y, z) = [x+2*y-z, y+z, x+y-2*z]
T = linear_transformation(U, U, h)      What this means? Define the linear transformation T : Q^3 → Q^3 by
print T.kernel()                                                   T(x) = h(x,y,z)

x1=vector([1, 2, 0]); x2=vector([1, 1, 1]); x3=vector([2,0,1])
x0=vector([1, 5 ,2])
C=column_matrix([x1, x2, x3])
y1=vector([4, -1, 3]); y2=vector([5, 5, 2]); y3=vector([6, 3, 3])
D=column_matrix([y1, y2, y3])
aug=D.augment(C, subdivide=True)
Q=aug.rref()
print Q

[G,mu]=A.gram_schmidt()     # G-S
B=matrix([G.row(i)/G.row(i).norm() for i in range(0,4)]); B
print B
print A._transpose()
print A.H    # same as Finding A.conjugate_transpose()

print A.jordan_form()


var('x y')
f=x^2+4*x*y+4*y^2+6*x+2*y-25
implicit_plot(f==0, (x,-10,10), (y,-10,10))

# var('t')                     # Define variables
# x=2+2*t                      # Define a parametric eq.
# y=-3*t-2
# parametric_plot((x,y), (t, -10, 10), rgbcolor='red')  # Plot
```

[ANSWER: output]

```
[ 2 -4  2  2 -1]
[-2  0  1  3  0]
[-2 -2  3  3  1]
[-2 -6  3  7  2]
```

What is this? Augmented matrix of the linear system $A\mathbf{x}=\mathbf{y}$

```
[ 2   0   0   8  : -8 ]
[ 0   2   0   6  : -6 ]
[ 0   0   1  11  : -8 ]
[ 0   0   0  16  : -11]
```

64 **What is this?** Determinant of A

```
[  1/2    1/2   -1/4   -1/4]
[  1/4    5/8  -1/16  -5/16]
[  1/4    1/8   7/16  -5/16]
[  1/4    5/8  -5/16  -1/16]
```

What this means? Inverse matrix of A

x^4 - 12*x^3 + 52*x^2 - 96*x + 64 **What is this?** Characteristic polynomial of A

[4, 4, 2, 2] **What is this?** Eigenvalues of A are 4 and 2.

[(**4**, [(0, 1, 1, 1)], **2**), (**2**, [(1, 0, -1, 1), (0, 1, 2, 0)], **2**)]

What are the algebraic and geometric multiplicities of each eigenvalues?

Algebraic multiplicities of $\lambda_1 = 4$ are 2 and geometric multiplicities is 1.

Algebraic multiplicities of $\lambda_1 = 2$ are 2 geometric multiplicities is 2.

Vector space of degree 3 and dimension 1 over Rational Field **What this means?** Dimension of Ker T is 1

Basis matrix:

[1 -1/3 1/3] **What this means?** Ker $T = \{t(1, -1/3, 1/3) : t \in \mathbb{R}\}$

```
[  1   0   0 | -1/2   3/2  -1/2 ]
[  0   1   0 |  0     2     -1  ]
[  0   0   1 | 1/2   -5/2   3/2 ]
```

What this means? Let $\alpha = \{x_1, x_2, x_3\}$ and $\beta = \{y_1, y_2, y_3\}$ be an ordered basis.

Then the transition matrix from β to α is $[I]_\beta^\alpha = \begin{bmatrix} -1/2 & 3/2 & -1/2 \\ 0 & 2 & -1 \\ 1/2 & -5/2 & 3/2 \end{bmatrix}$

```
[            1/7*sqrt(7)           -2/7*sqrt(7)            1/7*sqrt(7)             1/7*sqrt(7)]
[       -8/47*sqrt(94/7)        2/47*sqrt(94/7)        5/94*sqrt(94/7)       19/94*sqrt(94/7)]
[ -19/67*sqrt(134/47)   -7/67*sqrt(134/47)  53/134*sqrt(134/47)  -43/134*sqrt(134/47)]
[           -4*sqrt(1/67)          -5*sqrt(1/67)          -5*sqrt(1/67)            -sqrt(1/67)]
```

What is this?

The rows of matrix B are orthonormal. vectors by **Gram_Schmidt o.n. process.**

```
[ 2 -2 -2 -2 ]
[-4  0 -2 -6 ]
[ 2  1  3  3 ]
[ 2  3  3  7 ]
```

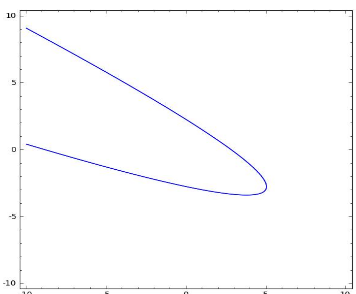
What is this? This matrix is A^T

```
[ 2 -2 -2 -2 ]
[-4  0 -2 -6 ]
[ 2  1  3  3 ]
[ 2  3  3  7 ]
```

What is this? This matrix is A^*

```
[ 4  | 1 | 0 | 0 ]
[ --  + -- + -- +-- ]
[ 0  | 4 | 0 | 0 ]
[ --  + -- + -- + --]
[ 0 |  0  | 2 | 0  ]
[ -- + -- + -- + -- ]
[ 0 |  0 |  0  | 2 ]
```

What is this? This matrix is the Jordan Canonical form of A



What this means? The graph of quadratic curve f is parabola.

V. (5pt x 4 = 20pts) Give a sketch of proof : <Choose and mark only 4 of 6!!>

1. **Show the set $\{1, x, x^2, \cdots, x^n\}$ is a linearly independent vectors** in a polynomial space composed of all of polynomial that have real coefficient and its degree is less than or equal to n.

2. How you are going to explain that if $A\mathrm{v} = \lambda\mathrm{v}$ and $\mathrm{v} \neq 0$, then

   λ^k is an eigenvalue corresponding to the eigenvector v of A^k for any positive integer k.

3. Let the columns of $A \in M_{m\times n}$ be linearly independent. Show that the set of column vectors in A^TA form a basis for $\mathbb{R}^n$.

4. If W_1 and W_2 are subspaces of a vector space V, show that $W_1 \cap W_2$ is a subspace of V.

5. Define a function $< , > : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by $<\mathrm{u},\mathrm{v}> = \mathrm{u}^T A \mathrm{v}$ when $A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$. **Then the function is an inner product.**

6. (Choose only one) **Explain how you prove the Schur's Theorem**

   (or How to find the Jordan Canonical Form from the given Dot Diagram.)

Your Answers:

1. Sketch of Proof (Definition of L.I.)

If $\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0$ for all x, (i.e. $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0$)

=> $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$ ($\because$ if not $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = 0) \neq 0$) (3 pt)

$\therefore \{1, x, x^2, \cdots, x^n\}$ is a set of linearly independent vectors. (2 pt) ■

[Another Method]

$$W(x_0) = \begin{vmatrix} f_1(x_0) & \cdots & f_n(x_0) \\ f_1'(x_0) & \cdots & f_n'(x_0) \\ \vdots & \vdots & \vdots \\ f_1^{(n-1)}(x_0) & \cdots & f_n^{(n-1)}(x_0) \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & \cdots & x^{n-1} \\ 0 & 0 & 2 & \cdots & x^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & n! \end{vmatrix} \neq 0$$ (is not always zero, not zero for some x)

$\therefore \{1, x, x^2, \cdots, x^n\}$ is a set of linearly independent vectors. (2 pt) (by Wronski's Test,)

2. Sketch of Proof [Show $A^k\mathrm{v} = \lambda^k\mathrm{v}$]

$A\mathrm{v} = \lambda\mathrm{v}$ => $A^2\mathrm{v} = A(A\mathrm{v}) = \lambda A\mathrm{v} = \lambda^2\mathrm{v}$ => $\cdots$ => $A^k\mathrm{v} = A(A^{k-1}\mathrm{v}) = \lambda^{k-1}A\mathrm{v} = \lambda^k\mathrm{v}$ ■

(Note: some of your proof using $P^{-1}AP = D$ may not working since the given matrix may not be diagonalizable.)

3. Sketch of Proof Hint : $\mathrm{rank}(A) = \mathrm{rank}(A^TA)$

Since A^TA is an $n\times n$ matrix and $\mathrm{rank}(A) = n = \mathrm{rank}(A^TA)$ ($\because$ the n column vectors of $A \in M_{m\times n}$ are linearly independent)

=> the n column vectors of $A^TA \in M_{n\times n}$ are linearly independent vectors in $\mathbb{R}^n$

=> the set of n column vectors in A^TA is linearly independent and spans $\mathbb{R}^n$.

=> the set of column vectors in A^TA form a basis for $\mathbb{R}^n$. ■

4. Sketch of Proof Let x, y $\in W_1 \cap W_2$ and k be a scalar. (2 step sub space test)

Since x, y $\in W_1$ and W_1 is the subspace of V => $\mathrm{x}+\mathrm{y} \in W_1$ and $k\mathrm{x} \in W_1$. (2 pt)

Since x, y $\in W_2$ and W_2 is the subspace of V => $\mathrm{x}+\mathrm{y} \in W_2$ and $k\mathrm{x} \in W_2$. (1 pt)

$\therefore$ $\mathrm{x}+\mathrm{y} \in W_1 \cap W_2$ and $k\mathrm{x} \in W_1 \cap W_2$ (2 pt)

Hence $W_1 \cap W_2$ is a subspace of V. ■

5. Sketch of Proof Let $\mathrm{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathrm{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\mathrm{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $c \in \mathbb{R}$.

(1) $<\mathrm{u},\mathrm{v}> = \mathrm{u}^TA\mathrm{v} = 3u_1v_1 + 2u_1v_2 + 2u_2v_1 + 4u_2v_2 = \mathrm{v}^TA\mathrm{u} = <\mathrm{v},\mathrm{u}>$

(2) $<\mathrm{u}+\mathrm{v},\mathrm{w}> = (\mathrm{u}+\mathrm{v})^TA\mathrm{w} = \mathrm{u}^TA\mathrm{w} + \mathrm{v}^TA\mathrm{w} = <\mathrm{u},\mathrm{w}> + <\mathrm{v},\mathrm{w}>$

(3) $<c\mathrm{u},\mathrm{v}> = c\mathrm{u}^TA\mathrm{v} = c<\mathrm{u},\mathrm{v}>$

<Part (4)> [Show $<\mathrm{u},\mathrm{u}> \geq 0$ and $<\mathrm{u},\mathrm{u}> = 0$ if and only if $\mathrm{u} = 0$]

$< \mathbf{u},\mathbf{u}> = \mathbf{u}^T A\mathbf{u} = \mathbf{u}^T \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{u} = [u_1 \ : \ u_2] \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 3u_1^2 + 4u_1u_2 + 4u_2^2 = (u_1+2u_2)^2 + 2u_1^2 \geq 0$ and

$(u_1+2u_2)^2 + 2u_1^2 = \mathbf{u}^T A\mathbf{u} = <\mathbf{u},\mathbf{u}> = 0 \Leftrightarrow u_1 + 2u_2 = 0 = u_1 \Leftrightarrow u_1 = u_2 = 0 \Leftrightarrow \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. ■

Hence the function $< \ , \ >$ is an inner product.

6. Sketch of Proof

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. We prove this by mathematical induction. First, if $n=1$, then the statement holds because $A = [\lambda_1]$. We now assume that the statement is true for any square matrix of order less than or equal to $n-1$.

① Let $\mathbf{x}_1$ be an eigenvector corresponding to eigenvalue λ_1.

② By the Gram-Schmidt Orthonormalization, there exists an orthonormal basis for $\mathbb{C}^n$ including $\mathbf{x}_1$, say $S = \{\mathbf{x}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n\}$.

③ Since S is orthonormal, the matrix $U_0 \equiv [\mathbf{x}_1 \ : \mathbf{z}_2 : \ \cdots \ : \mathbf{z}_n]$ is a unitary matrix. In addition, since $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, the first column of AU_0 is $\lambda_1\mathbf{x}_1$. Hence $U_0^*(AU_0)$ is of the following form:

$$U_0^* A U_0 = \left[\begin{array}{c:c} \lambda_1 & * \\ \hdashline O & A_1 \end{array}\right]$$

where $A_1 \in M_{n-1}(\mathbb{C})$. Since $|\lambda I_n - A| = (\lambda - \lambda_1)\,|\lambda I_{n-1} - A_1|$, the eigenvalues of A_1 are $\lambda_2, \lambda_3, \ldots, \lambda_n$.

④ By the induction hypothesis, there exists a unitary matrix $\widehat{U_1} \in M_{n-1}(\mathbb{C})$ such that

$$\widehat{U_1}^* A_1 \widehat{U_1} = \begin{bmatrix} \lambda_2 & & * \\ 0 & & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

⑤ Letting $U_1 \equiv \left[\begin{array}{c:c} 1 & 0 \ \cdots \ 0 \\ \hdashline 0 & \\ \vdots & \widehat{U_1} \\ 0 & \end{array}\right] \in M_n(\mathbb{C})$, we get

$$(U_0U_1)^* A (U_0U_1) = U_1^* U_0^* A U_0 U_1 = \begin{bmatrix} \lambda_1 & & & * \\ & & & \\ 0 & \lambda_2 & & * \\ 0 & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Since $U \equiv U_0U_1$ is a unitary matrix, the result follows. ■

(or Sketch) For $A \in M_n(\mathbb{C})$, let r_j denote the number of dots in the jth row of the dot diagram of λ_i. Then, the following are true.

(1) $r_1 = n - \mathrm{rank}(A - \lambda_i I)$.

(2) If $j > 1$, $r_j = \mathrm{rank}((A - \lambda_i I)^{j-1}) - \mathrm{rank}((A - \lambda_i I)^j)$.

For a 9×9 matrix A_i, the number of Jordan blocks contained in A_i is l and the size of the Jordan blocks is completely determined by $p_1, p_2, \ldots, p_l$. To see this, take $l = 4$ and $p_1 = 3, p_2 = 3, p_3 = 2, p_4 = 1$. Then, following the sequence of block sizes,

$$A_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

is uniquely determined. To find the dot diagram of A_i, since $l = 4$, $p_1 = 3$, $p_2 = 3$, $p_3 = 2$ and $p_4 = 1$, the dot diagram of is:

• • • •   (Number of Jordan blocks: 4)

• • •

• •

(The End)