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Cauchy's Interlace Theorem for Eigenvalues of Hermitian Matrices

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Hermitian matrices have real eigenvalues. The Cauchy interlace theorem states that the eigenvalues of a Hermitian matrix A of order n are interlaced with those of any principal submatrix of order $n-1$.

Theorem 1 (Cauchy Interlace Theorem). *Let A be a Hermitian matrix of order n, and let B be a principal submatrix of A of order $n-1$. If $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_2 \le \lambda_1$ lists the eigenvalues of A and $\mu_n \le \mu_{n-1} \le \cdots \le \mu_3 \le \mu_2$ the eigenvalues of B, then $\lambda_n \le \mu_n \le \lambda_{n-1} \le \mu_{n-1} \le \cdots \le \lambda_2 \le \mu_2 \le \lambda_1$.*

Proofs of this theorem have been based on Sylvester's law of inertia [**3**, p. 186] and the Courant-Fischer minimax theorem [**1**, p. 411], [**2**, p. 185]. In this note, we give a simple, elementary proof of the theorem by using the intermediate value theorem.

Proof. Simultaneously permuting rows and columns, if necessary, we may assume that the submatrix B occupies rows $2, 3, \ldots, n$ and columns $2, 3, \ldots, n$, so that A has the form

$$A = \begin{bmatrix} a & \mathbf{y}^* \\ \mathbf{y} & B \end{bmatrix},$$

where $*$ signifies the conjugate transpose of a matrix. Let $D = \operatorname{diag}(\mu_2, \mu_3, \ldots, \mu_n)$. Then, since B is also Hermitian, there exists a unitary matrix U of order $n-1$ such that $U^*BU = D$. Let $U^*\mathbf{y} = \mathbf{z} = (z_2, z_3, \ldots, z_n)^T$.

We first prove the theorem for the special case where $\mu_n < \mu_{n-1} < \cdots < \mu_3 < \mu_2$ and $z_i \ne 0$ for $i = 2, 3, \ldots, n$. Let

$$V = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & U \end{bmatrix},$$

in which $\mathbf{0}$ denotes the zero vector. Then V is a unitary matrix and

$$V^*AV = \begin{bmatrix} a & \mathbf{z}^* \\ \mathbf{z} & D \end{bmatrix}.$$

Let $f(x) = \det(xI - A) = \det(xI - V^*AV)$, where I denotes the identity matrix. Expanding $\det(xI - V^*AV)$ along the first row, we get

$$f(x) = (x-a)(x-\mu_2)\cdots(x-\mu_n) - \sum_{i=2}^{n} f_i(x),$$

where $f_i(x) = |z_i|^2(x-\mu_2)\cdots(\widehat{x-\mu_i})\cdots(x-\mu_n)$ for $i = 2, 3, \ldots, n$. Note that $f_i(\mu_j) = 0$ when $j \neq i$ and that

$$f_i(\mu_i) \begin{cases} > 0 & \text{if } i \text{ is even,} \\ < 0 & \text{if } i \text{ is odd,} \end{cases}$$

hence that

$$f(\mu_i) \begin{cases} < 0 & \text{if } i \text{ is even,} \\ > 0 & \text{if } i \text{ is odd,} \end{cases}$$

for $i = 2, 3, \ldots, n$. Since $f(x)$ is a polynomial of degree n with positive leading coefficient, the intermediate value theorem ensures the existence of n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $f(x) = 0$ such that $\lambda_n < \mu_n < \lambda_{n-1} < \mu_{n-1} < \cdots < \lambda_2 < \mu_2 < \lambda_1$.

For the proof of the general case, let $\epsilon_1, \epsilon_2, \ldots$ be a sequence of positive real numbers such that $\epsilon_k \downarrow 0$, $z_i + \epsilon_k \neq 0$ for $i = 2, 3, \ldots, n$ and $k = 1, 2, \ldots,$ and the diagonal entries of $D + \epsilon_k \operatorname{diag}(2, 3, \ldots, n)$ are distinct for fixed k. For $k = 1, 2, \ldots,$ let

$$C_k = \begin{bmatrix} a & \mathbf{z}(\epsilon_k)^* \\ \mathbf{z}(\epsilon_k) & D(\epsilon_k) \end{bmatrix},$$

where $\mathbf{z}(\epsilon_k) = \mathbf{z} + \epsilon_k(1, 1, \ldots, 1)^T$ and $D(\epsilon_k) = D + \epsilon_k \operatorname{diag}(2, 3, \ldots, n)$, and let $A_k = VC_kV^*$. Then A_k is Hermitian and $A_k \to A$. Let $\lambda_n^{(k)} \leq \lambda_{n-1}^{(k)} \leq \cdots \leq \lambda_2^{(k)} \leq \lambda_1^{(k)}$ list the eigenvalues of A_k. Then

$$\lambda_n^{(k)} < \mu_n + n\epsilon_k < \lambda_{n-1}^{(k)} < \mu_{n-1} + (n-1)\epsilon_k < \cdots < \lambda_2^{(k)} < \mu_2 + 2\epsilon_k < \lambda_1^{(k)}.$$

Since $\lambda_n^{(k)}, \lambda_{n-1}^{(k)}, \ldots, \lambda_1^{(k)}$ are n distinct roots of $\det(xI - A_k) = 0$ for each k and since the graph of $y = \det(xI - A_k)$ is sufficiently close to that of $y = \det(xI - A)$, it follows that $(\lambda_n^{(k)}, \lambda_{n-1}^{(k)}, \ldots, \lambda_1^{(k)}) \to (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ and the proof is complete.

■

If β is an eigenvalue of B of multiplicity p, then by Theorem 1 we see that β is an eigenvalue of A of multiplicity $p-1$ or p or $p+1$. The following question arises naturally: *When is β an eigenvalue of A of multiplicity p or larger?* An answer to this question is obtained by making a simple observation concerning the characteristic polynomial $f(x) = \det(xI - A)$ of A.

Suppose that $\beta = \mu_2 = \mu_3 = \cdots = \mu_{p+1} > \mu_{p+2}$. Then

$$\begin{aligned} f(x) = {} & (x-a)(x-\beta)^p(x-\mu_{p+2})\cdots(x-\mu_n) \\ & - \left(\sum_{i=2}^{p+1} |z_i|^2\right)(x-\beta)^{p-1}(x-\mu_{p+2})\cdots(x-\mu_n) \\ & - \sum_{i=p+2}^{n} |z_i|^2(x-\beta)^p(x-\mu_{p+2})\cdots(\widehat{x-\mu_i})\cdots(x-\mu_n). \end{aligned}$$

Let $h(x)$ be the polynomial such that $f(x) = (x - \beta)^{p-1}h(x)$. Then $h(\beta) = 0$ if and only if $\sum_{i=2}^{p+1} |z_i|^2 = 0$ or, equivalently, if and only if $z_i = 0$ for $i = 2, 3, \ldots, p+1$. Let $U = [\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n]$ be the unitary matrix such that $U^*BU = D$. Then, since $z_i = \mathbf{u}_i^*\mathbf{y}$ and since the vectors $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{p+1}$ form a basis for the eigenspace of B corresponding to the eigenvalue β, we have the following conclusion.

Theorem 2. *Let*

$$A = \begin{bmatrix} a & \mathbf{y}^* \\ \mathbf{y} & B \end{bmatrix}$$

be a Hermitian matrix, and let β be an eigenvalue of B of multiplicity p. Then β is an eigenvalue of A of multiplicity at least p if and only if $\mathbf{y}$ is orthogonal to the eigenspace of B corresponding to the eigenvalue β.

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New Designs for the Descartes Rule of Signs

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The question of how to construct polynomials having as many roots as allowed by the Descartes rule of signs has been the focus of interest recently [**1**], [**2**]. For a given real polynomial

$$p(x) = a_0 + a_1x + \cdots + a_nx^n,$$

Descartes's rule of signs says that the number of positive roots of $p(x)$ is equal to the number of sign changes in the sequence $a_0, a_1, \ldots, a_n$, or is less than this number by a positive even integer. Investigating which of the possible numbers of roots permitted by Descartes actually occur, Anderson, Jackson, and Sitharam [**1**] show that the rule cannot be improved. For any sign sequence not containing zero, they construct polynomials with this sign sequence in the coefficients and any prescribed number of positive roots that is in accord with Descartes's rule. Analogously, since the negative roots are the roots of $p(-x)$, all allowable numbers of negative roots are realized. Grabiner [**2**] establishes further examples. In particular, he provides polynomials for sign sequences that may contain zero and shows how to achieve certain numbers of positive and negative roots simultaneously.