Fall 2013, Calculus II, Final Comprehensive Exam					Sign
Course	**Calculus II** with Sage	GEDB021-42	Prof.	Sang-Gu Lee	

Major		Year(학년)		S.N(학번)		Name	

※ Notice (수험생 유의사항)

1. Write your name and get the professor's signature. 답안 작성전에 이름 등을 빠짐없이 기입하고 감독자 날인을 받으세요.
2. Keep your Honor code. If not, a serious penalty will be given.

I (2*12=24pts)	II (4*6=24)	III (4*10=40)	VI (6*2=12)	Total 100

Score (200)	
참여+발표+PBL	EXAM
/100	/100

I. (2pt x 12 = 24) Mark True(T) or False(F) in the blanks ( ).

1. (F) ***Squeeze (Sandwich) Property:*** If $a_n \le b_n \le c_n$ is true **for some** n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

2. (F) A series $\sum a_n$ is absolutely convergent if $\sum a_n$ converges, and $\sum |a_n|$ diverges. **(if $\sum |a_n|$ converges)**

3. (F) If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$, then $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. ($\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$)

4. (T) Suppose $f(x,y)$ and it's derivatives f_x, f_y, f_{xy} and f_{yx} are defined in a domain containing a point (a,b) and all are continuous at (a,b). Then $f_{xy}(a,b) = f_{yx}(a,b)$.

5. (F) Let $z = f(x,y)$ have continuous first order partial derivatives and $y = g(x)$ be a differentiable function of x. Then the composition $z = f(x, g(x))$ is a differentiable function of x and $\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dy} + \frac{\partial z}{\partial y}\frac{dy}{dx}$. ($\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$)

6. (T) Given a function f differentiable at (a, b), the line tangent to the level curve of f at (a, b) is orthogonal to the gradient $\nabla f(a,b)$.

7. (T) Let f be a continuous function on a polar rectangular region D given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$ where $0 \le \beta - \alpha \le 2\pi$. Then
$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \int_a^b \int_\alpha^\beta f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr.$$

8. (T) Let $\phi : V \subseteq \mathbb{R}^3 \to W \subseteq \mathbb{R}^3$ given by $\phi(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ be one-one and onto function having the 1st order continuous partial derivatives. We further assume that the Jacobian $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \ne 0$. Let $f : W \to \mathbb{R}$ be a continuous function. Then $\iiint_W f(x,y,z)\,dV = \iiint_V f[x(u,v,w), y(u,v,w), z(u,v,w)] \left|\frac{\partial(x,y,w)}{\partial(u,v,w)}\right| du\,dv\,dw$.

9. (F) Suppose we write a two dimensional vector field in the form $\mathbf{F} = (M, N, 0)$ where M and N are functions of x and y. Then
$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\mathbf{k}, \text{ and so } (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \langle 0, 0, N_x - M_y\rangle \cdot \langle 0, 0, 1\rangle = N_x - M_y. \text{ Thus}$$
Green's Theorem say $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Mdx - Ndy = \iint_D (N_x - M_y)\,dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA$. **($\int_C Mdx + Ndy$)**

10. (T) Let S be a smooth surface given the vector valued function $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in R^2$ and f be a continuous function. Then the surface integral of the function f over S is
$$\iint_S f(x,y,z)\,dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dA.$$

11. (T) Let f be a scalar function and F a vector field. curl (grad f) is a vector field.

12. (F) Let f be a scalar function and F a vector field. grad (div F) is a scalar function. (vector field.)

Ⅱ. (4pt x 6 = 24) State or Define. Let f be a scalar function and $\mathbf{F}$ a vector field.

1. Choose 2 terminologies or concepts from each group (**A, B, C**) and state their meanings as much as you can.

A. ① Integral Test ② p-series Test ③ Comparison Test ④ Limit Comparison Test ⑤ Alternating Series Test ⑥ Ratio Test ⑦ Root Test ⑧ Taylor series and Maclaurin series ⑨ Gamma function : gamma (2, 1) ⑩ Arc Length ⑪ Curvature ⑫ Limit of a function with two variables ⑬ Cross product of two vectors ⑭ Cauchy-Schwarz Inequality ⑮ Triangle Inequality ⑯ Projection ⑰ Tangential and normal components of the acceleration vector, $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\lvert\mathbf{r}'(t)\rvert}$ and $a_N = \dfrac{\lvert\mathbf{r}'(t)\times\mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert}$ ⑱ curvature

1.

The Comparison Test. Suppose that $0 \leqslant a_n \leqslant b_n$ for sufficiently large n.

- If $\sum a_n$ diverges, then $\sum b_n$ also diverges.
- If $\sum b_n$ converges, then $\sum a_n$ also converges.

(일부 답안 제시, 나머지는 교재 참조)

2.

Definition 2. [1] The *Taylor series* for $y = f(x)$ at x_0 is the power series:

$$P_\infty(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad \text{(open form)}$$

B. ① cylindrical coordinate, spherical coordinate ② level curves and level surfaces ③ meaning of grad f ④ directional derivative ⑤ conservative ⑥ potential function ⑦ curl and divergence ⑧ Lagrange multipliers

3. Let $\mathbf{F}$ be a vector field. Then $\mathbf{F}$ is called conservative if there is a scalar function f such that $\text{grad} f = \mathbf{F}$.

A function f is called a potential function (or scalar potential) for $\mathbf{F}$.

4.

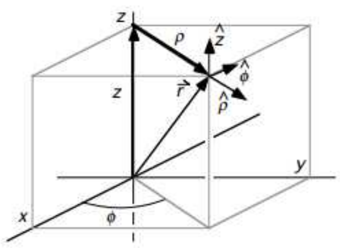
Cylindrical Coordinates

Transforms

The forward and reverse coordinate transformations are

$$\rho = \sqrt{x^2 + y^2} \qquad x = \rho\cos\phi$$
$$\phi = \arctan(y, x) \qquad y = \rho\sin\phi$$
$$z = z \qquad z = z$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.



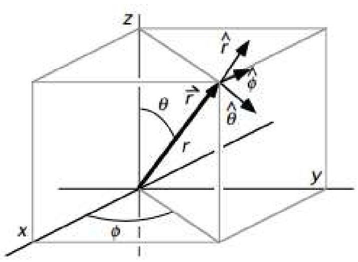
Spherical Coordinates

Transforms

The forward and reverse coordinate transformations are

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\sin\theta\cos\phi$$
$$\theta = \arctan\left(\sqrt{x^2 + y^2}, z\right) \qquad y = r\sin\theta\sin\phi$$
$$\phi = \arctan(y, x) \qquad z = r\cos\theta$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.



C. ① Fubini's Theorem ② Green's Theorem ③ Stokes' Theorem ④ Gauss Divergence Theorem

5. [**Green's Theorem**] If R is a closed region in the xy−plane bounded by a simple closed curve C and if $M(x, y)$ and $N(x, y)$ are continuous functions of x and y having continuous first order partial derivatives in R, then $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$ where C is traversed in the positive direction.

6. [**Stokes' Theorem**] Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field that components have continuous second order partial derivatives in a domain containing a surface S bounded by a simple closed curve C. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S}$ where C is traversed in the positive direction.

[**Gauss Divergence Theorem**] Let S be a closed surface in $\mathbb{R}^3$ which is the boundary of a solid region V. Let $\mathbf{A}$ be a vector field defined on some open set T containing V. Suppose the components of $\mathbf{A}$ have continuous first order partial derivatives and $\mathbf{n}$ is an unit normal vector that is directed outward from S. Then

$$\iint_S \mathbf{A} \cdot \mathbf{n}\,dS = \iiint_V \nabla \cdot \mathbf{A}\,dV = \iiint_V \text{div}\,\mathbf{A}\,dV.$$

Ⅲ. (2pt x 19 = 38pt) Find or Explain or Fill the blanks.

1. Use $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, $|x|<1$ to find a power series representation for $f(x)=\ln(1-x^2)$. What is the radius of convergence?

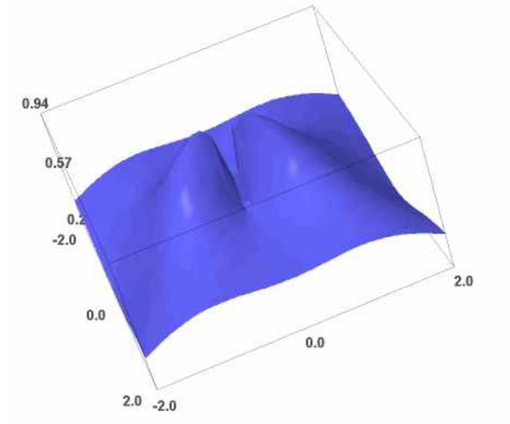
Sol $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$, $|x|<1$

$\Rightarrow \ln(1-x^2) = -\sum_{n=1}^{\infty}\frac{(x^2)^n}{n}$, $|x|<1$.

The radius of convergence R = 1 ■

2. Explain why $\lim_{(x,y)\to(0,0)}\frac{x^2+\sin^2 y}{2x^2+y^2}$ **does not exist.**

```
var('x, y')
f(x, y)=(x^2+sin(y)^2)/(2*x^2+y^2)
plot3d(f(x, y), (x, -2, 2), (y, -2, 2))
```



Sol ① $x=0$, $y\to 0$(즉 y축을 따라 원점으로 접근):

$$\lim_{(x,y)\to(0,0)}\frac{x^2+\sin^2 y}{2x^2+y^2} = \lim_{\substack{x=0\\y\to 0}}\frac{0^2+\sin^2 y}{2\cdot 0^2+y^2} = \lim_{y\to 0}\frac{\sin^2 y}{y^2} = 1$$

② $y=0$, $x\to 0$(즉 x축을 따라 원점으로 접근):

$$\lim_{(x,y)\to(0,0)}\frac{x^2+\sin^2 y}{2x^2+y^2} = \lim_{\substack{y=0\\x\to 0}}\frac{x^2+\sin^2 0}{2x^2+0^2} = \lim_{x\to 0}\frac{x^2}{2x^2} = \frac{1}{2}$$

① 과 ②의 경로를 따라 원점으로 다가갈 때 극한값이 다르므로 극한이 존재하지 않는다.

■

3. Find the velocity, acceleration and speed of a particle with the given position function $\mathrm{r}(t) = <2t^2+1,\ t^3,\ 2t^2-1>$.

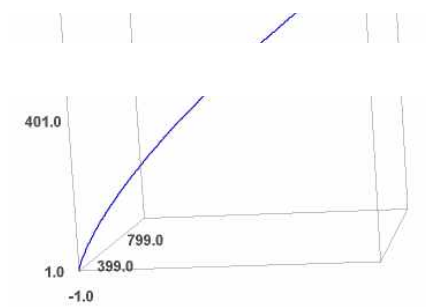
Sol $\mathrm{v}(t) = \mathrm{r}'(t) = \quad <4t,\ 3t^2,\ 4t>$

$\Rightarrow \mathrm{a}(t) = \mathrm{r}''(t) = <4,\ 6t,\ 4>$

$\Rightarrow \mathrm{s}(t) = |\mathrm{v}(t)| = \sqrt{(4t)^2+(3t^2)^2+(4t)^2} = \sqrt{9t^4+32t^2}$

http://math1.skku.ac.kr/home/pub/1355/

```
r(t)= (2*t^2 + 1, t^3, 2*t^2 - 1)
v= diff(r(t), t)          #v는 속도벡터함수, diff는 미분하는 명령어
a= diff(v(t), t)          #a는 가속도벡터함수
s= v.norm()               #s는 속도의 크기, .norm()은 벡터의 크기를 구한다.
p1 = parametric_plot3d(r(t), (t, 0, 20), color='blue', width= 2)
show (p1)
r(t); v; a; s
```



Position Vector : (2*t^2 + 1, t^3, 2*t^2 - 1)
Velocity Vector : (4*t, 3*t^2, 4*t)
Acceleration Vector : (4, 6*t, 4)

Speed : $\sqrt{9t^4+32t^2}$ ■

4. (a) Change from cylindrical coordinates $(2, -\frac{\pi}{3}, 4)$ to rectangular coordinates.

```
T = Cylindrical('height', ['radius', 'azimuth'])
T.transform(radius=2, azimuth=- pi/3, height=4)
```

Answer : $(1, -\sqrt{3}, 4)$

(b) Change from spherical coordinates $(5, \frac{\pi}{3}, \frac{\pi}{2})$ to rectangular coordinates.

```
T = Spherical('radius', ['azimuth', 'inclination'])
T.transform(radius=5, azimuth=pi/3, inclination=pi/2)
```

Answer: $(\frac{5}{2}, \frac{5}{2}\sqrt{3}, 0)$ ■

5. Find the volume of the sphere with radius a.

Sol The sphere $S = \{x^2+y^2+z^2 \le a^2\}$ can be described in spherical coordinates as $\{(\rho, \phi, \theta) | 0 \le \rho \le a, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$.

Then, $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{3}\rho^3 \sin\phi\right]_0^a d\phi\, d\theta$

$= \frac{1}{3}a^3\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = \frac{1}{3}a^3\int_0^{2\pi}[-\cos\phi]_0^{\pi} d\theta = \frac{2}{3}a^3\int_0^{2\pi} d\theta = \frac{4}{3}\pi a^3.$ ■

6. Find $\iiint_E (x+1)\, dV$, where $E = \{(x,y,z) |\ 0 \le x \le 1,\ 0 \le y \le x,\ -y^2 \le z \le x^2\}$.

Sol http://matrix.skku.ac.kr/cal-lab/cal-14-5-3(new).html

```
var('x,y,z')
f(x,y)=integral(x+1,(z,-y^2,x^2))
g(x)=integral(f(x,y),(y,0,x))
integral(g,(x,0,1))
```

Answer : 3/5

$\iiint_E (x+1)\, dV = \int_0^1\int_0^x\int_{-y^2}^{x^2} (x+1)\, dz\, dy\, dx$

$= \int_0^1\int_0^x [(x+1)z]_{-y^2}^{x^2} dy\, dx$

$= \int_0^1\int_0^x (x+1)(x^2+y^2)\, dy\, dx$

$= \int_0^1\left[(x+1)\left(x^2y + \frac{1}{3}y^3\right)\right]_0^x dx = \frac{4}{3}\int_0^1 (x^4+x^3)\, dx = \frac{3}{5}$ ■

7. Find the equation of normal line: 점 $(1, 1, \frac{\pi}{4})$에서 곡면 $z = f(x,y) = \tan^{-1}\frac{x}{y}$에 접하는 접평면과 법선의 방정식을 구하여라.

Sol $f_x(x,y) = \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y} = \frac{y}{x^2+y^2} \Rightarrow f_x(1,1) = \frac{1}{2}$

$f_y(x,y) = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2} \Rightarrow f_y(1,1) = -\frac{1}{2}$

이므로 접평면의 방정식은 $\frac{1}{2}(x-1) - \frac{1}{2}(y-1) - (z - \frac{\pi}{4}) = 0$, 즉 $x - y - 2z + \frac{\pi}{2} = 0$이다.

법선의 방정식은 $\frac{x-1}{\frac{1}{2}} = \frac{y-1}{-\frac{1}{2}} = \frac{z-\frac{\pi}{4}}{-1}$, 즉, $2(x-1) = -2(y-1) = -(z-\frac{\pi}{4})$ 이다. ■

8. Find a partial derivative $\frac{\partial z}{\partial y}$: 방정식 $x+2y+z-2\ln(xyz)=0$에 의해 결정되는 음함수의 편도함수 $\frac{\partial z}{\partial y}$를 구하여라.

Sol $F(x, y, z) = x+2y+z-2\ln(xyz) = 0$이라 하자.

$F_x = 1-2\frac{yz}{xyz} = 1-\frac{2}{x} = \frac{x-2}{x}$,

$F_y = 2-2\frac{xz}{xyz} = 2-\frac{2}{y} = \frac{2y-2}{y}$,

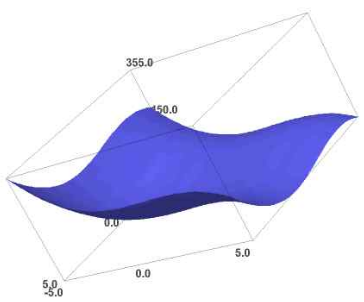
$F_z = 1-2\frac{xy}{xyz} = 1-\frac{2}{z} = \frac{z-2}{z}$ 에서

$$\therefore \ \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{x-2}{x}}{\frac{z-2}{z}} = -\frac{z(x-2)}{x(z-2)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{2y-2}{y}}{\frac{z-2}{z}} = -\frac{2z(y-1)}{y(z-2)}$$ ■

9. 함수 $f(x,y)=x^3-y^3+3x^2+3y^2-9x$에 대하여 임계점을 모두 구하고 그 점에서 극대, 극소값을 갖는지 혹은 안장점인지를 판별하여라.

Sol http://matrix.skku.ac.kr/Cal-Book/part2/CS-Sec-13-8-Sol.html

```
var('x, y')
f(x, y)=x^3-y^3+3*x^2+3*y^2-9*x
plot3d(f(x, y), (x, -5, 5), (y, -5, 5))
```




```
var ('x, y, z')
f(x,y)= x^3-y^3+3*x^2+3*y^2-9*x
fx=f.diff(x)
fy=f.diff(y)
fxx=diff(f,x,x)
fyy=diff(f,y,y)
fxy=diff(f,x,y)
cpoints=solve([fx==0,fy==0],[x,y],solution_dict=True)
for sol in cpoints:
    if ((sol[x] in RR) and (sol[y] in RR) ):
        print((sol[x],sol[y]))
```

```
def extreme(f,a,b):
    f11=diff(f,x,x)(a,b)
    f22=diff(f,y,y)(a,b)
    f12=diff(f,x,y)(a,b)
    D=f11*f22-f12^2
    if(D>0):
        if(f11>0):
            return "local minimum"
        else:
            if(f11<0):
                return "local maximum"
            else:
                return "inconclusive"
    else:
        if(D<0):
            return "saddle point"
        else:
            if(D==0):
                return "inconclusive"
table = [["Critical Point", "Type"]]
f(x,y)= x^3-y^3+3*x^2+3*y^2-9*x
fx=f.diff(x)
fy=f.diff(y)
fxx=diff(f,x,x)
fyy=diff(f,y,y)
fxy=diff(f,x,y)
cpoints=solve([fx==0,fy==0],[x,y],solution_dict=True)
for sol in cpoints:
    if ((sol[x] in RR) and (sol[y] in RR)):
        a=sol[x].n()
        b=sol[y].n()
        table.append([(sol[x],sol[y]), extreme(f,a,b)])
html.table(table, header=True)
```

$f_x(x,y) = 3x^2+6x-9=0 \quad \Rightarrow \quad x=-3, 1$

$f_y(x,y) = -3y^2+6y=0 \quad \Rightarrow \quad y=0, 2$

따라서 임계점 $(-3,0), (-3,2), (1,0), (1,2)$를 얻는다.

$f_{xx}(x,y)=6x+6$, $f_{xy}=0$, $f_{yy}=-6y+6$이므로 각점에 대하여 다음과 같이 판별하면

점 point	$A=f_{xx}$	$B=f_{xy}$	$C=f_{yy}$	$D=AC-B^2$
$(-3,0)$	-12	0	6	$-$
$(-3,2)$	-12	0	-6	$+$
$(1,0)$	12	0	6	$+$
$(1,2)$	12	0	-6	$-$

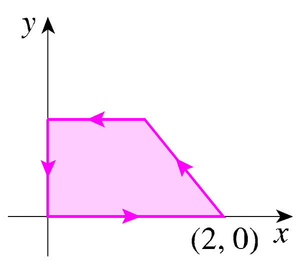
Critical Point	Type
$(-3,0)$	saddle point
$(1,0)$	local minimum
$(-3,2)$	local maximum
$(1,2)$	saddle point

Answer : $f(x,y)=x^3-y^3+3x^2+3y^2-9x$ 는 극소값 -5 과 극대값 31을 갖는다.

10. [Green's Theorem] Let C be the straight lines from $(0,0)$ to $(2,0)$, from $(2,0)$ to $(1,1)$, from $(1,1)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$. Evaluate $\int_C e^{x^2}dx+xydy$.

Sol Let $P(x, y)=e^{x^2}$, $Q(x, y)=xy$. Then

$$\int_C P(x, y)dx+Q(x, y)dy = \iint_D \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} dA = \iint_D y dA = \int_0^1 \int_0^{2-y} y \, dx \, dy = \frac{3}{2}.$$ ■



IV. (3+5= 8pt) Prove or Explain (Fill the blanks).

1. Show that the function defined by $f(x,y)=\begin{cases} \dfrac{xy}{2x^2+3y^2}, & when\ (x,y)\neq(0,0) \\ 0 & when\ (x,y)=(0,0) \end{cases}$ is not continuous at $(0,0)$

but its first order partial derivatives exist at $(0,0)$.

▶ Sol

먼저 $f(x,y)$가 $(0,0)$에서 연속이 아님을 보이자.

① $x=0,\ y\to 0$(즉 y축을 따라 원점으로 접근):

$$\lim_{(x,y)\to(0,0)}\frac{xy}{2x^2+3y^2}=\lim_{\substack{x=0\\y\to 0}}\frac{0\cdot y}{2\cdot 0^2+3y^2}=\lim_{y\to 0}\frac{0}{3y^2}=0$$

② $y=x,\ x\to 0$(즉 직선 $y=x$를 따라 원점으로 접근):

$$\lim_{(x,y)\to(0,0)}\frac{xy}{2x^2+3y^2}=\lim_{\substack{y=x\\x\to 0}}\frac{x\cdot x}{2x^2+3x^2}=\lim_{x\to 0}\frac{x^2}{5x^2}=\frac{1}{5}$$

이므로, ① 과 ②의 경로를 따라 원점으로 다가갈 때 극한값이 다르므로 극한이 존재하지 않는다. 따라서 연속도 아니다.

그런데, $f_x(0,0)=\lim_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{\frac{h\cdot 0}{2h^2+3\cdot 0^2}}{h}=0$

$f_y(0,0)=\lim_{h\to 0}\dfrac{f(0,0+h)-f(0,0)}{h}=\lim_{h\to 0}\dfrac{\frac{0\cdot h}{2\cdot 0^2+3h^2}}{h}=0$ 이므로 $(0,0)$에서 $f(x,y)$의 편도함수는 존재한다.

2. Find $\dfrac{\partial z}{\partial v}$ when $z=x^2\ln y,\ x=\dfrac{v}{u},\ y=u^2+v^2$.

▶ Sol

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=2x\ln y\cdot\left(\frac{1}{u}\right)+\frac{x^2}{y}\cdot 2v \quad =\frac{2v\ln(u^2+v^2)}{u^2}+\frac{2v^3}{u^2(u^2+v^2)}$$

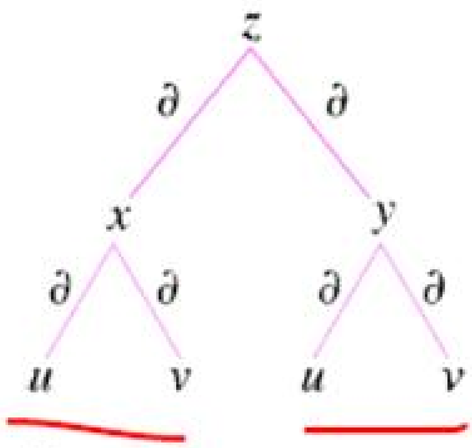
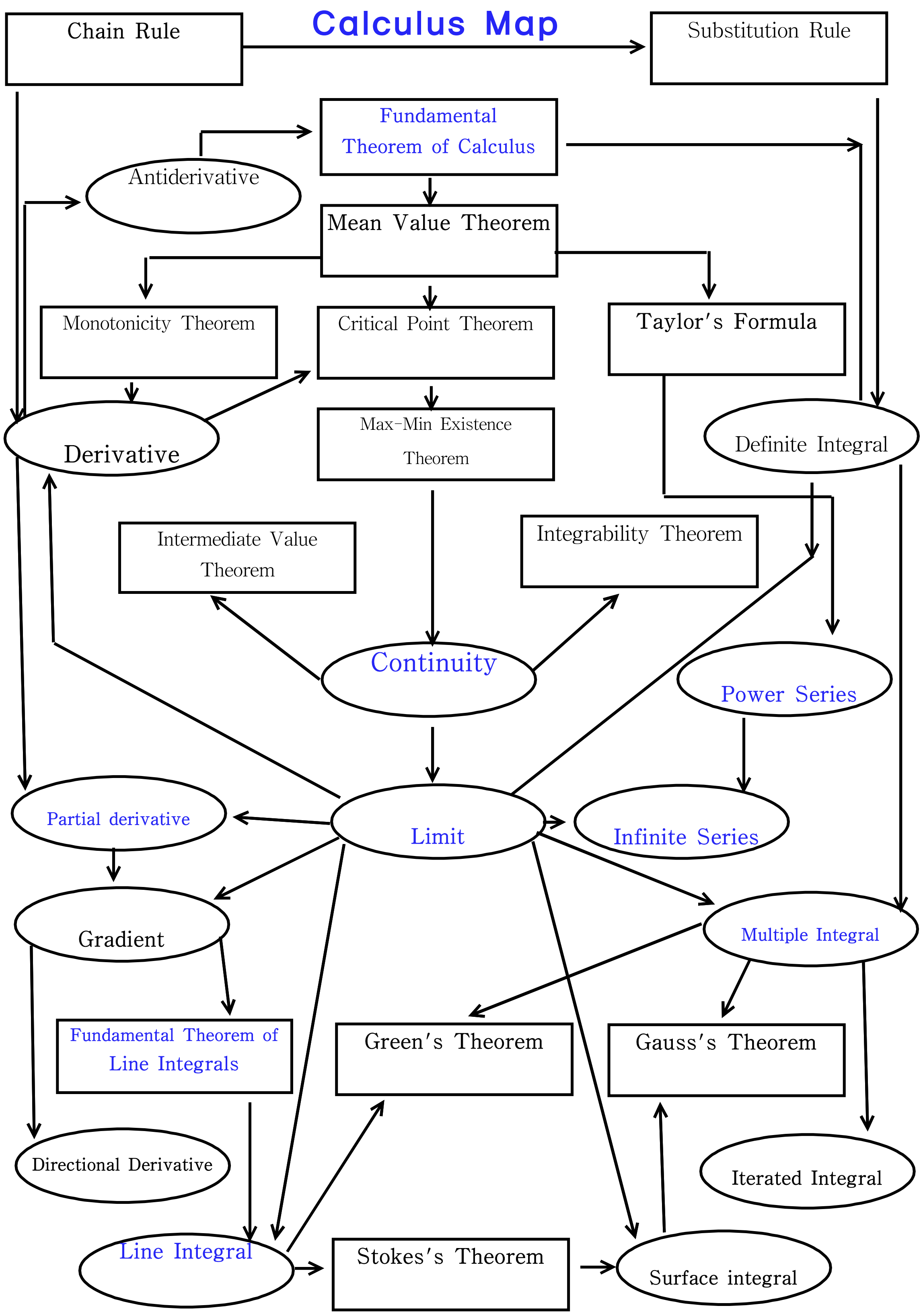


Figure 3

■ The End. Have a nice winter break!

Appendix 1



Calculus Map
Chain Rule
Substitution Rule
Fundamental Theorem of Calculus
Antiderivative
Mean Value Theorem
Monotonicity Theorem
Critical Point Theorem
Taylor's Formula
Derivative
Max-Min Existence Theorem
Definite Integral
Intermediate Value Theorem
Integrability Theorem
Continuity
Power Series
Partial derivative
Limit
Infinite Series
Gradient
Multiple Integral
Fundamental Theorem of Line Integrals
Green's Theorem
Gauss's Theorem
Directional Derivative
Iterated Integral
Line Integral
Stokes's Theorem
Surface integral

-- Appendix 2 --

http://matrix.skku.ac.kr/Cal-Book/ http://sagenb.skku.edu/ http://matrix.skku.ac.kr/Mobile-Sage-G/sage-grapher.html
http://sage.skku.edu/ http://sage.knou.ac.kr/(http://101.79.75.132/) http://sage.knou.ac.kr/ (http://101.79.75.157)

```
var('t')                                  # 변수정의 (매개변수방정식 )
x=2+2*t; y=-3*t-2
parametric_plot((x, y), (t, -10, 10), rgbcolor='red')  # 직선 Plot
var('a, b, c, d, x, y, z, r, n') # 변수정의 (매개변수방정식)
f(x)=exp(-2*x)                            # 함수 정의
g(x)=f.taylor(x,2,4)                      # 테일러전개
plot(f(x), (x, -2, 2))                    # 함수의 그래프
implicit_plot(sin(x)-2*y==3, (x, -2, 2), (y, -2, 2)) # 음함수 그래프
solve(f(x)==0, x)                 # Solve 방정식 풀이
diff(f(x), x, 2)                  # f(x)의 2계 도함수
integral(f(x), x)                 # x에 관한 부정적분
integral(f(x), x, -2, 3)          # [-2, 3]까지의 정적분
limit(f(x), x=2)                  # 극한
limit(f(x), x=2, dir='+')         # 우극한
limit(f(x), x=2, dir='-')         # 좌극한
limit(f(x), x=+oo)                # +infinity 에서의 극한
limit(f(x), x=-oo)                # -infinity 에서의 극한
limit(ln(3*n)/ln(5*n), n=+oo)
a(n) = ((e)/10)^n
sum(a(n), n, 1, +oo)  # 급수의 계산

u(n)=1/(n*2^n)
rho=limit(abs(u(n+1)/u(n)), n=+oo)
R=1/rho; R                        # 수렴반경

x = (3, -4, 5)
a=sqrt(x[0]^2)             #  yz평면과의 거리
b=sqrt(x[1]^2)             #  zx평면과의 거리
c=sqrt(x[2]^2)             #  xz평면과의 거리
d=sqrt(x[1]^2 +x[2]^2)     #  x 축과의 거리
e=sqrt(x[0]^2 +x[2]^2)     #  y 축과의 거리
f=sqrt(x[0]^2 +x[1]^2)     #  z 축과의 거리

f.partial_fractions(x)            # 부분분수
find_root(f(x), a, b)             # [a, b] 사이에서 근사해 구하기
a=vector(QQ, [2, -4, 3]);d= -2
a.norm()                          # 벡터의 크기
distance=abs(a.dot_product(a)+d)/a.norm()    # 거리
r(t)=(t-t^3, t^2, 0)        # 벡터함수 정의
dr=diff(r(t), t)
ddr=diff(r(t), t, 2)
T=dr.dot_product(ddr)/dr.norm()      #.dot_product()는 내적
N=(dr.cross_product(ddr)).norm() / dr.norm()  #.cross_product()는 외적
var('t')                    # 변수정의 (매개변수방정식)
r(t)=(t-t^3, t^2, 0)        # 벡터함수 정의
dr=diff(r(t),t)
g(t)=sqrt((dr[0]^2+dr[1]^2+dr[2]^2).simplify_trig());
k=integral(g(t),t,-5,5)       # arc length
S=integral(r(t), t)           # integral은 적분하는 명령어
```

```
v=diff(r(t), t)              # diff는 t에 관해 도함수 구하는 명령어
a=diff(v, t)                 # diff(r(t), t, 2) 와 a는 가속도 벡터함수
s= v.norm()       # s는 속도의 크기, .norm()은 벡터의 크기를 구한다.
v2=v.subs(t=2)          # v2는 t=2 일 때 속도
a2=a.subs(t=2)          # a2는 t=2 일 때 가속도
p1=parametric_plot(r(t), (t, 0, 3)) # parametric_plot은 매개변수함수 그림
p2=line([r(2), r(2)+v2], color='red') # line은 선분을 두 점을 잇는 선분
p3=line([r(2), r(2)+a2], color='green')
show(p1+p2+p3)
var('x, y, z')
p1=implicit_plot3d(x^2+y^2==5, (x, -3, 3), (y, -3, 3),  (z, -3, 3))
p2=implicit_plot3d(x*y==z, (x, -3, 3), (y, -3, 3), (z, -3, 3), rgbcolor='green', opacity=0.4)
show(p1+p2)
contour_plot(f(x, y), (x, -1, 1), (y, -1, 1), fill=False, cmap='hsv', axes=True, labels=True)  # level curve 그리기

var('x, y, z')
f=x*y+y*z^2+x*z^3
f.gradient()                         # Find gradient
integral(integral(f, x, 0, y), y, 0, 1)   # 이중적분
integral(integral(integral(f, x, 0, y), y, 0, 1),z,0,1)    # 삼중적분
var('r,theta')
f=arctan(tan(theta))
integral(integral(f*r,r,1,2),theta,0,pi/4)  # 이중적분 (극좌표)

T = Cylindrical('height', ['radius', 'azimuth']) #cylindrical ->rectangular
T.transform(radius=1, azimuth= pi, height=2)
T = Spherical('radius', ['azimuth', 'inclination']) #spherical->rectangular
T.transform(radius=3, azimuth=pi/6, inclination=pi/6)

T=Cylindrical('radius', ['azimuth', 'height'])
theta, z=var('theta, z')
plot3d(3*cos(theta), (theta, 0, 2*pi), (z, -2, 2), transformation=T)

vf=plot_vector_field((x+y,x), (x,-3,3), (y,-3,3), aspect_ratio=1) #벡터필드

plot_vector_field3d((0,0,1), (x, -3,3), (y,-3,3), (z,-3,3)) #3차원 벡터필드

integral(f.dot_product(diff(r,t)),t,0,pi) # Line integral

A(x,y,z) = P*i+Q*j+R*k   # A의 curl, conservative if curl(F)=0.
curlA=(diff(R,y)-diff(Q,z))*i+(diff(P,z)-diff(R,x))*j+(diff(Q,x)-diff(P,y))*k
divA = diff(P,x)+diff(Q,y)+diff(R,z)    # A 의 divergence

integral(integral((diff(N,x)-diff(M,y))*r, r, 0, 3), t, 0, 2*pi) # Green 정리
integral(integral(curl(F).dot_product(-n), r, 0, 1),t, 0, 2*pi) # Stokes 정리
integral(integral(integral(Div,0,3),y,0,2),z,0,1) # Divergence 정리
```