Spring 2013, Calculus I, Midterm Exam (Solution)						Sign
Course	**Calculus I** with Sage		GEDB020-42	Prof.	Sang-Gu Lee	
Major		Year(학년)		S.N(학번)		Name

※ **Notice (수험생 유의사항)**

1. Write your name and get the signature. 답안 작성전에 이름 등을 빠짐없이 기입하고 감독자 날인을 받으세요.
2. Keep your Honor code. If not, serious penalty will be given.

Score (150)	
PBL	EXAM
/ 50	/100

http://matrix.skku.ac.kr/cal-lab/sage-grapher.html , http://matrix.skku.ac.kr/cal-lab/sage-grapher-para.html

var('a,b,c,d')	# 변수정의	integral(f(x), x)	# 부정적분
limit(f(x), x=a)	# 극한	integral(f(x), x, a, b)	# 정적분
limit(f(x), x=a, dir='minus')	# 좌극한	plot(f(x), (x, a, b)	# 함수의 그래프
limit(f(x), x=a, dir='plus')	# 우극한	implicit_plot(f, (x, a, b), (y, c, d))	# 음함수 그래프
limit(f(x), x=+oo)	# 무한대에서의 극한	find_root(f(x), a, b)	# 근사해 구하기
limit(f(x), x=-oo)		var('t')	# 변수정의 (매개변수방정식)
solve(f(x)==0, x)	# Solve 방정식 풀이	x=2+2*t	
diff(f(x), x)	# 도함수	y=-3*t-2	
diff(f(x), x, 2)	# 2계 도함수	parametric_plot((x,y), (t, -10, 10), rgbcolor='red')	# 직선 Plot

I. (2pt x 12 = 24) Mark True(T) or False(F) in the blank ( ).

1. (F) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ also ~~exists~~. (Conterexample : 0/0)

2. (T) If $f(x)$ has a local minimum (or maximum) at $x=c$ and $f'(c)$ exists, then $f'(c)=0$.

3. (T) Let f be continuous on $[a,b]$. Suppose $a < f(a) < b$ and $a < f(b) < b$, then there exists a number c in (a, b) such that $f(c)=c$.

4. (T) If $f(x) \le g(x)$ for all $a \le x \le b$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

5. (F) If $\int_a^b f(x)dx = 0$, then ~~$f(x)=0$ on $[a,b]$~~. (Conterexample : $f(x)=x$ on $[-1, 1]$)

6. (T) Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \ne f(b)$. Then there exists a number c in (a, b) such that $f(c)=N$.

7. (F) The most famous application of the use of hyperbolic cosine is to describe the shape of a hanging wire. If a heavy flexible cable (such as a telephone or power line) is suspended between two points at the same height, then it takes the shape of **a catenary curve** with equation ~~$y=c+a\sinh(x/a)$~~, which uses the hyperbolic cosine. (The Latin word catena means **"chain."**) ($y=c+a\cosh(x/a)$)

8. (**T**) Let f be a function whose second derivative exists on an open interval I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

9. (F) Let $C(x)$ be the cost function of producing x units of a certain product. Then the marginal cost is the rate of change of $C(x)$ with respect to x, that is, the marginal cost function is $C'(x)$. The marginal cost function is ~~$C'(x)$~~ represents the cost per unit when x units are produced. ($\frac{C(x)}{x}$)

10. (F) If $f(x)$ is ~~not~~ continuous on $[a,b]$, then there is ~~no~~ $\xi \in [a,b]$ such that $\int_a^b f(x)\,dx = f(\xi)(b-a)$.

11. (F) If f is ~~not~~ a continuous function on $[a,b]$, then $F(x) = \int_a^x f(t)\,dt$ is ~~not~~ continuous on $[a,b]$ and $F(x)$ is ~~not~~ differentiable on (a,b).

12. (F) Let f be a continuous function on $[a,b]$. Suppose that F is continuous on $[a,b]$ and that $F'=f$ on (a,b). Then ~~$\int_a^b f(x)dx = |F(b)-F(a)|$~~.

($\int_a^b f(x)dx = F(b)-F(a)$)

Ⅱ. (3pt x 5= 15) State or Define.

1. Choose 3 terminology or concept and state its meaning as much as you knows.

Extreme Value theorem, Intermediate Value theorem, Fermat' s theorem, Mean Value theorem, Squeeze theorem, Fundamental Theorem of Calculus, [differential(미분), dy], [linear approximation(선형근사)], [Riemann Sum(리만합)], [convexity(볼록성)], [inflection point(변곡점)], [Chain Rule(연쇄법칙)], [Differentiation of Implicit function(음함수의 미분법)]

▶▶ **Sol** **(1)** [differential(미분), dy] Given a function $y=f(x)$ we call dy and dx **differentials** and the relationship between them is given by, $dy=f'(x)dx$. Also $f'(x)dx$ will be called **a differential of** $f(x)$ and denoted by dy or $df(x)$. Note that if we are just given $f(x)$ then the differentials are df and dx and we compute them the same manner. $df=f'(x)dx$

(2) [the Squeeze Theorem (or Sandwich Theorem)]

If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

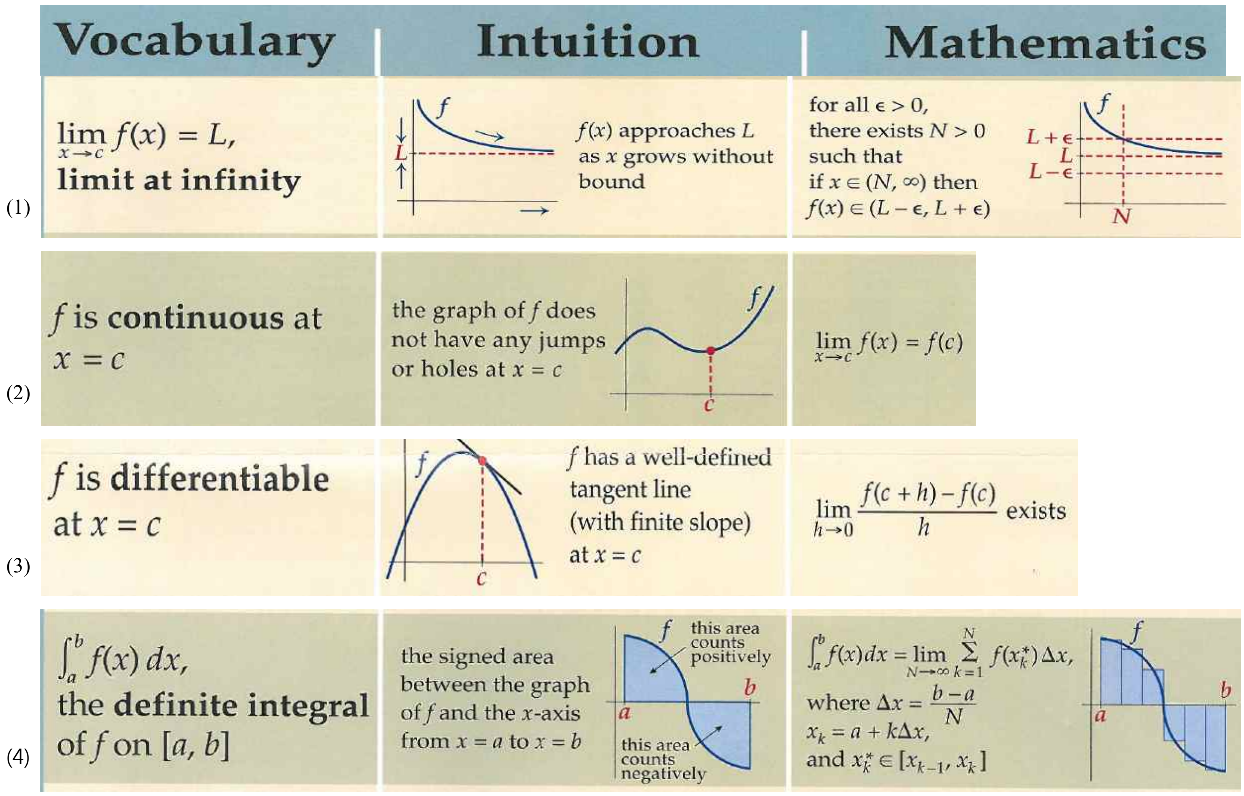
(3) ...

2. **Discuss a geometric and physical interpretation of $f'(x)$.**

▶▶ **Sol** **Geometric interpretation :** The slope of the tangent to the curve at a point is equal to the derivative of the function at that point.

Physical interpretation : The average velocity is the ratio between distance travelled (Δd) and the time elapsed (Δt). ■

3. **Choose 1 definition, and state in your words what it means.**



▶▶ **Sol** (1) Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then $\lim_{x\to a} f(x) = \infty$ means that for every positive number M there is a positive number δ such that $f(x) > M$ whenever $0 < |x-a| < \delta$.

(2) A function f is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$. So if a function f is continuous at $x=a$, then f will satisfy the following three conditions : 1. $f(a)$ is defined (that is, a is in the domain of f) 2. $\lim_{x\to a} f(x)$ exists 3. $\lim_{x\to a} f(x) = f(a)$

(3) A function f is differentiable at a if $f'(a)$ exists.

(4) ...

4. State the Procedure for Newton's Method.

Sol Let us consider the graph of $y=f(x)$ and we want to solve $f(x)=0$. We start with the (proper, 해에 충분히 가까운) initial approximation x_1, which may be obtained by just guessing, or examining the graph of f. Then we use the tangent line L to the curve $y=f(x)$ at the point $(x_1, f(x_1))$ to approximate the curve and look at the x-intercept of L, labeled x_2. The equation of the tangent line L is

$y-f(x_1)=f'(x_1)(x-x_1)$. Thus, we obtain $0-f(x_1)=f'(x_1)(x_2-x_1)$. If $f'(x_1)\neq 0$, we can solve this equation for x_2: $x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$

Under certain conditions, x_2 is usually a better approximation to the solution than x_1. Then we repeat this procedure with x_1 replaced by x_2, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation: $x_3=x_2-\dfrac{f(x_2)}{f'(x_2)}$. Continuing this process obtains a sequence of approximations x_1, x_2, x_3, ... as shown in the Figure. In general, if $f'(x_n)\neq 0$ then we have $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$. The number x_n becomes closer and closer to the solution if the sequence $\{x_n\}$ converges as $n\to\infty$. We note that if $f'(x_1)\to 0$ then the sequence may not converge. In this case, we have to choose a different initial x_1. ■

5. State what you know about the number $e=\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=1+\dfrac{1}{1}+\dfrac{1}{1\cdot 2}+\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots \approx 2.71828$:

Sol The number e is an important mathematical constant, approximately equal to 2.71828, that is the base of the natural logarithm. This number arises in the study of compound interest, and can also be calculated as the sum of the infinite series $e=1+\dfrac{1}{1}+\dfrac{1}{1\cdot 2}+\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots$. The constant can be defined in many ways; for example, e is the unique real number such that the value of the derivative (slope of the tangent line) of the function $f(x)=e^x$ at the point $x=0$ is equal to 1. The number $e=2.71828182845905\ldots$ is defined so that when $a=e$ from $f(x)=a^x$ as $x\to 1$. There is a very important exponential function that arises naturally in many places. This function is called the natural exponential function. However, for most people this is simply the exponential function. For $f(x)=e^x$, $\lim_{h\to 0}\dfrac{e^h-1}{h}=1$ implies $f'(0)=1$, and from the general derivative above we have $f'(x)=e^x$. Thus the slope of a tangent line to the curve $y=e^x$ is equal to the y-coordinate of the point. The Natural Exponential Function : $e=\lim_{x\to 0}(1+x)^{1/x}$, $e=\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ or $\lim_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n=e^x$.

Since the logarithm is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function, $\exp(x)=y \Leftrightarrow \ln y=x$ and $\exp(\ln x)=x$ and $\ln(\exp x)=x$. In particular, we have $\exp(0)=1$ since $\ln 1=0$. $\exp(1)=e$ since $\ln e=1$. We obtain the graph of $y=\exp(x)$ by reflecting the graph of $y=\ln x$ about the line $y=x$. The domain of exp is the range of the logarithm. That is, $(-\infty, \infty)$ which is the range of the logarithm, is the domain of the exponential and $(0,\infty)$, which is the domain of the logarithm is the range of exponential. If r is any rational number, then the third law of logarithms gives $\ln(e^r)=r\ln e=r$. Therefore, by $\exp(r)=e^r$. Thus, $\exp(x)=e^x$ whenever x is a rational number. This leads us to define e^x, even for irrational values of x, by the equation $e^x=\exp(x)$. In other words, for the reasons given, we define e^x to be the inverse of the function $\ln x$.

Properties of the Exponential Function. The exponential function $f(x)=e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0,\infty)$. Thus, $e^x>0$ for all x. Also $\lim_{x\to-\infty}e^x=0$ $\lim_{x\to\infty}e^x=\infty$. So the x-axis is a horizontal asymptote of $f(x)=e^x$. Exponential Function: Consider the exponential function $f(x)=a^x$ where $a>0$, $a\neq 1$.

From the definition of the derivative: $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{a^{x+h}-a^x}{h}=\lim_{h\to 0}\dfrac{a^x a^h-a^x}{h}=\lim_{h\to 0}\dfrac{a^x(a^h-1)}{h}$. Thus

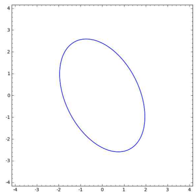
$f'(x)=a^x\lim_{h\to 0}\dfrac{a^h-1}{h}=a^x\lim_{h\to 0}\dfrac{f(h)-f(0)}{h}=a^x f'(0)$. That is, the rate of change of any exponential function is proportional to the function itself. Furthermore $\lim_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{1/n}=e^{-1}$ because $\ln S_n=\dfrac{1}{n}\left(\ln\dfrac{1}{n}+\ln\dfrac{2}{n}+\cdots+\ln\dfrac{n}{n}\right)$ and $\lim_{n\to\infty}\ln S_n=\lim_{n\to 0}\dfrac{1}{n}\ln\dfrac{k}{n}=\int_0^1\ln x\,dx=[x\ln x-x]_0^1$ $=-1-x\ln x=-1$ if $S_n=\left(\dfrac{n!}{n^n}\right)^{1/n}$.

e is the number such that $\ln e=1$. ■

Ⅲ. (3pt x 13 = 39pt) Find or Explain or Fill the blank.

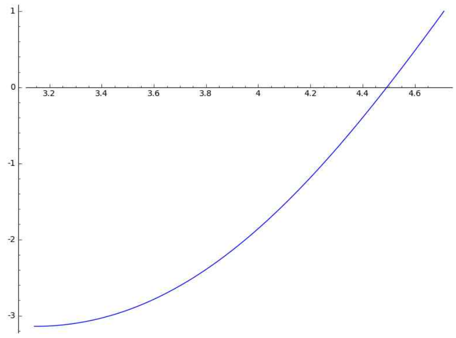
1. State the Sage command that plot the implicit function $7x^2+4xy+4y^2-23=0$ $(-4 \le x \le 4,\ -4 \le y \le 4)$.

```
var('x, y')
f = 7*x^2 + 4*x*y + 4*y^2-23
implicit_plot(f, (x, -4, 4), (y, -4,4))
```



2. The followings explain that the equation $\sin x = x\cos x$ has at least one real root on $[\pi, 3/2\pi]$ using Intermediate Value Theorem.

```
var('x')
f(x)=(x)*cos(x) - sin(x)
plot(f(x), x, pi, (3/2)*pi)
```



```
print f(pi), f((3/2)*pi)
```

Sol -pi, 1 ■

Since f(x)=(x)*cos(x) - sin(x) is continuous on $[\pi, 3/2\pi]$ and $f(\pi) < 0 < f((3/2)*\pi)$, using Intermediate Value Theorem, there exist at least one real root of the equation $\sin x = x\cos x$ on $[\pi, 3/2\pi]$.

```
find_root(f(x), pi, (3/2)*pi)
```

Sol 4.493409457909064 ■

3. Find $f(0)$, which make $f(x)=(1+x^2)^{-\frac{5}{x^2}}$ be continuous at $x=0$. [Hint: Use $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$]

Sol If $f(x)$ is continuous at $x=0$, $f(0)=\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+x^2)^{-\frac{5}{x^2}} = \lim_{x\to 0}[(1+x^2)^{\frac{1}{x^2}}]^{-5} = e^{-5}$.

$\therefore\ f(0)=e^{-5}$.

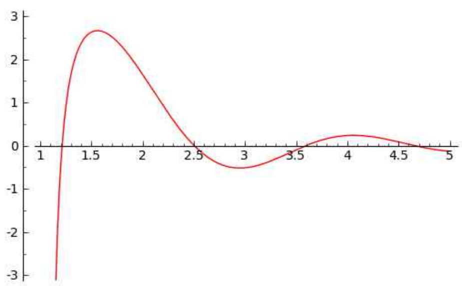
Sage 명령어

```
x=var('x')                      # 변수 지정
limit((1+x^2)^(-5/(x^2)), x=0)  # e^-5
```

■

4. We plot the graph of the derivative of $f(x)=\dfrac{\sin 3x}{x(x-1)}$ $(1 \le x \le 5)$ using Sage below.

```
var('x')                                              # 변수 지정
f(x)=(sin(3*x))/(x*(x-1))                             # 함수 입력
df(x)=diff(f(x), x)                                   # 도함수 계산
plot(df(x), (x, 1, 5), ymax=3, ymin=-3, color='red')  # 도함수 그래프 그리기
```



The graph intersects the x-axis at $x=1.2,\ 2.5,\ 3.6,\ 4.7$.

(1) Find intervals on which f is decreasing.

Sol $f'(x)<0$ 이 되는 구간에서 함수 $f(x)$가 감소하므로 위 그래프에서 $f'(x)$가 x 축 아래에 위치하는 범위를 찾으면 된다.

∴ 감소구간 : $[1, 1.2]$, $[2.5, 3.6]$, $[4.7, 5]$ ■

(여기서 끝점은 포함하지 않아도 된다. 즉, 열린구간으로 써도 된다.)

(2) Find x at which $f(x)$ has local extreme values

Sol $f'(x)$가 존재하지 않거나 $f'(x)=0$이 되는 critical points (임계점)의 좌우에서 도함수 $f'(x)$의 값이 $(+) \to (-)$ 이면 극대, $(-) \to (+)$ 이면 극소가 된다.
위의 그래프를 통해 살펴보면

∴ 극댓값을 가지는 x의 값은 2.5, 4.7. 극솟값을 가지는 x의 값은 1.2, 3.6이다. ■

5. $\lim_{x\to 0}\frac{[f(x)-f(0)]\sin 2x}{x^2}=4$. Find $f'(0)$ [Hint: Use the definition of $f'(0)$, $\lim_{x\to 0}\frac{\sin x}{x}=1$, the properties of limits.]

Sol $4=\lim_{x\to 0}\frac{[f(x)-f(0)]\sin 2x}{x^2}=\lim_{x\to 0}\frac{f(x)-f(0)}{x}\cdot\frac{\sin 2x}{2x}\cdot 2$

$=2\left(\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\right)\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)=2f'(0)\cdot 1=2f'(0)$ ($\because$ the definition of $f'(0)$)

$\therefore\ f'(0)=2$ ■

6. Find the limit using natural logarithm and L'Hospital's Rule

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$$

Sol Let $y=(\cos x)^{\frac{1}{x^2}}$ $\Rightarrow$ $\ln y=\frac{1}{x^2}\ln(\cos x)=\frac{\ln(\cos x)}{x^2}$

$\Rightarrow\ \lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(\cos x)}{x^2}=\lim_{x\to 0}\frac{-\frac{\sin x}{\cos x}}{2x}=-\frac{1}{2}\lim_{x\to 0}\frac{\tan x}{x}=-\frac{1}{2}\lim_{x\to 0}\frac{\sec^2 x}{1}$ ($\because$ L'Hospital's Rule)

$=-\frac{1}{2}\lim_{x\to 0}\sec^2 x=-\frac{1}{2}$

$\Rightarrow\ \ln(\lim_{x\to 0}y)=\lim_{x\to 0}\ln y=-\frac{1}{2}$ and $\lim_{x\to 0}y=e^{-\frac{1}{2}}$ ($\because$ continuous ft.)

$\therefore\ \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}=e^{-\frac{1}{2}}$ ■

7. The tangent line at $(0,\frac{\pi}{2})$ of parametric equation $x=t\cos t,\ y=t\sin t,\ t>0$ is $y-\frac{\pi}{2}=\frac{-2}{\pi}(x-0)$ since $\left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\frac{\pi}{2}}=-\frac{2}{\pi}$.

Find the velocity (속도) and speed (속력) at $t=\frac{\pi}{2}$.

Sol **At $(0,\frac{\pi}{2})$, the velocity (속도) = $\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\left(-\frac{\pi}{2},1\right)$** and **speed (속력) = $\sqrt{\frac{\pi^2}{4}+1}$** ■

8. Use differential to approximate $\sqrt{98}$.

Sol Let $f(x)=\sqrt{x}$. Set $x=100$ and $\Delta x=-2$. Since $df\approx\Delta x$, $df=\frac{dx}{2\sqrt{x}}$, we have $df|_{x=100}=-\frac{1}{10}$.

Hence approximately, $\sqrt{98}=f(100+\Delta x)\approx f(100)+df|_{x=100}=\sqrt{100}-\frac{1}{10}=\frac{99}{10}=9.9$. ■

9. A closed cylindrical can is to hold $500cm^3$ of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

Sol $\pi r^2 h=500$ and $h=\frac{500}{\pi r^2}$

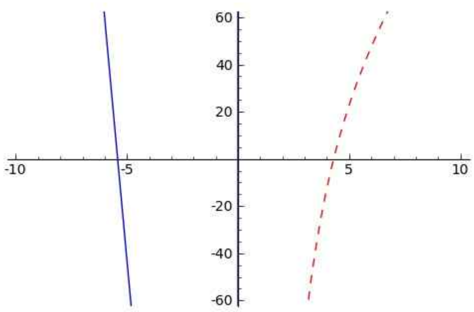
$S=2\pi r^2+2\pi rh=2\pi r^2+2\frac{500}{r}$

$\Rightarrow\ S'=4\pi r-\frac{1000}{r^2}$

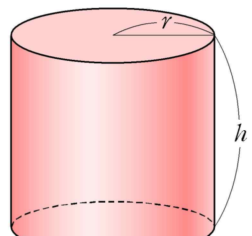
Let $S'=0$

$\Rightarrow r=\sqrt[3]{\frac{250}{\pi}}$ and $h=\frac{500}{\pi(250/\pi)^{2/3}}=10\left(\frac{2}{\pi}\right)^{1/3}$ ■

http://matrix.skku.ac.kr/cal-lab/cal-4-4-exs-5.html




```
var('x,y');
f=2*pi*x^2 + 1000/x
P= plot(f,x, xmin=-10, xmax=10, ymin=-60, ymax=60)
df=diff(f, x)
Q=plot(df, x, xmin=-10, xmax=10, ymin=-60, ymax=60,linestyle="--", color='red')
solve(df==0, x)   # (250/pi)^(1/3)
numerical_approx( (250/pi)^(1/3), 100) # 4.3
show(P+Q)
```



10. If $P(x)$ is the total value of the production when there are x workers in a plant, then the average productivity is $A(x) = \frac{P(x)}{x}$.

Find A'. Explain why the company wants to hire more worker if $A'(x) > 0$?

Sol $A'(x) = \frac{P'(x)x - P(x)}{x^2}$

If $A'(x) > 0$, then $P'(x)x - P(x) > 0$ $(\because\ x^2 > 0)$

$P'(x)$ is the rate of productivity.

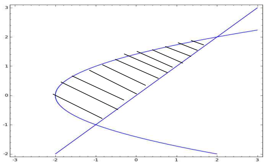
$P'(x) - \frac{P(x)}{x} > 0$ $(\because\ x > 0)$

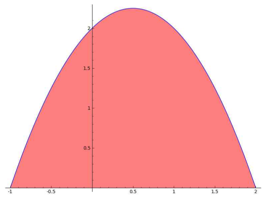
$P'(x) > \frac{P(x)}{x}$

This means the rate of productivity $P'(x)$ is larger than the average productivity $A(x) = \frac{P(x)}{x}$ which means if the company hire more workers, then they can expect to have a better productivity. ■

11. Evaluate the area covered by $y^2 = x + 2$ and $x - y = 0$.

Sol Consider $x = y$ and $x = y^2 - 2$. Then the area is





$\int_{-1}^{2} \{y - (y^2 - 2)\}dy$

$$\int_{-1}^{2} \{y - (y^2 - 2)\}dy = \int_{-1}^{2} (y - y^2 + 2)dy$$

$$= \left[\frac{1}{2}y^2 - \frac{1}{3}y^3 + 2y\right]_{-1}^{2}$$

$$= \frac{4}{2} - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2 = \frac{9}{2}$$

Sage :

```
var('y')
integral(y-y^2 + 2, y, -1, 2)
```

Answer : 9/2 ■

12. $\frac{\sin x}{1 + x\sin x}$ is an anti-derivative of $f(x)$. Find $\int f(x)f'(x)dx$. [Hint: Substitute $u = f(x)$ and $du = f'(x)dx$]

$$f(x) = \left(\frac{\sin x}{1 + x\sin x}\right)' = \frac{\cos x(1 + x\sin x) - \sin x(\sin x + x\cos x)}{(1 + x\sin x)^2} = \frac{\cos x - \sin^2 x}{(1 + x\sin x)^2}$$

$$\int f(x)f'(x)dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\{f(x)\}^2 + C = \frac{(\cos x - \sin^2 x)^2}{2(1 + x\sin x)^4} + C$$ ■

13. A honeybee population starts with 30 bees and increases at a rate of $n(t) = t^3 - 2t$ bees per week. How many honeybees are there after 10 weeks?

Sol Since the net change in population during 10 weeks is $\int_0^{10} n(t)dt = 2400$, the total number of honeybees after 10 weeks is $30 + 2400 = 2430$. ■

IV. (4pt x 4 = 16pt) Prove or Explain (Fill the blank).

1. $\lim_{x \to -1} x^2 + 2x - 8 = -9$

▶▶ **Sol** $\forall\ \epsilon > 0$ [Find δ] Let $\delta = \sqrt{\epsilon}$

If $0 < |x+1| < \delta$, then $|f(x) - (-9)| = |x^2 + 2x + 1| = |(x+1)^2| = |(x+1)|^2 < \delta^2 = \epsilon$.

[Side calculation] $f(x) - f(-1) = x^2 + 2x - 8 + 9 = x^2 + 2x + 1 = (x+1)^2$. ■

2. Show $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ implies $\frac{d}{dx}(\text{sech}\ x) = -\text{sech}\ x \tanh x$.

Proof : $\frac{d}{dx}(\text{sech}\ x) = (\frac{1}{\cosh x})' = \frac{0 - \sinh x}{(\cosh x)^2} = -\frac{\sinh x}{\cosh x}\frac{1}{\cosh x} = -\tanh x\,\text{sech}\ x$ ■

3. If f is a continuous function on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and it is differentiable on (a, b) and $F'(x) = f(x)$.

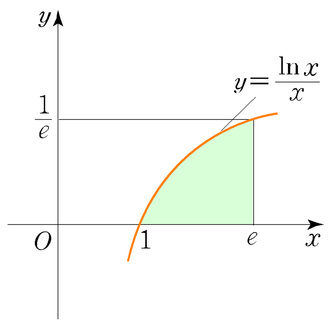
Proof : Let x be a point in (a, b). $\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\left\{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt\right\}$

$= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt$.

By Mean Value theorem for integration, there exist ξ in $[x, x+\Delta x]$ such that $\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt = f(\xi)$

Since $\xi \to x$ as $\Delta x \to 0$, and $f(x)$ is continuous. $F'(x) = \lim_{\Delta x \to 0}\frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0}\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt = \lim_{\xi \to x} f(\xi) = f(x)$. ■

4. Find $\int_1^e \frac{\ln x}{x} dx$.



Figure

▶▶ **Sol** The integrand suggests using $u = \ln x$, so then $du = dx/x$.
Now when $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln\ e = 1$.

Thus $\int_1^e \frac{\ln x}{x} dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$. ■

(QnA Participation, 4pt) Write one good example of your Note or Solution or Answer in QnA.

More than 400 problems were solved and revised and finalized in Q&A . I have made more than 4*7 contributions in it including ...
That changed my ...

(Bonus, 2pt) What you have newly learned and improved from our Honor Calculus with Sage?

Now I can draw, find and explain. And eventually I can solve most of problems in any calculus book by hand or by Sage! That was a difference.

******* [별지] **Honor Calculus** Spring 2013, Midterm Exam *********

http://matrix.skku.ac.kr/cal-lab/sage-grapher.html , http://matrix.skku.ac.kr/cal-lab/sage-grapher-para.html

var('a,b,c,d')	# 변수정의	integral(f(x), x)	# 부정적분
limit(f(x), x=a)	# 극한	integral(f(x), x, a, b)	# 정적분
limit(f(x), x=a, dir='minus')	# 좌극한	plot(f(x), (x, a, b)	# 함수의 그래프
limit(f(x), x=a, dir='plus')	# 우극한	implicit_plot(f, (x, a, b), (y, c, d))	# 음함수 그래프
limit(f(x), x=+oo)	# 무한대에서의 극한	find_root(f(x), a, b)	# 근사해 구하기
limit(f(x), x=-oo)		var('t')	# 변수정의 (매개변수방정식)
solve(f(x)==0, x)	# Solve 방정식 풀이	x=2+2*t	
diff(f(x), x)	# 도함수	y=-3*t-2	
diff(f(x), x, 2)	# 2계 도함수	parametric_plot((x,y), (t, -10, 10), rgbcolor='red')	# 직선 Plot

Derivatives of Inverse Trigonometric Functions[1]

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$$

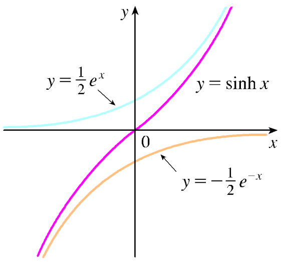
Definition of the Hyperbolic Functions[2]

$$\sinh x=\frac{e^x-e^{-x}}{2} \qquad \cosh x=\frac{e^x+e^{-x}}{2}$$

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}} \qquad \coth x=\frac{\cosh x}{\sinh x}=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

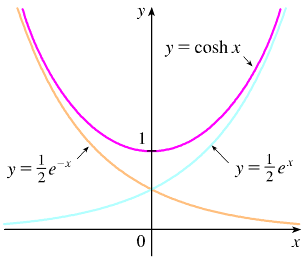
$$\operatorname{sech} x=\frac{1}{\cosh x}=\frac{2}{e^x+e^{-x}} \qquad \operatorname{csch} x=\frac{1}{\sinh x}=\frac{2}{e^x-e^{-x}}$$

The graphs of the hyperbolic sine and cosine can be sketched using graphical addition as in Figures 8 and 9.



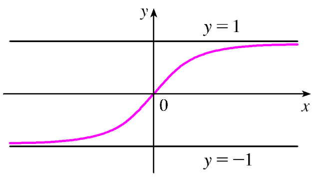
$y=\sinh x=\frac{1}{2}e^x-\frac{1}{2}e^{-x}$

Figure 8



$y=\cosh x=\frac{1}{2}e^x+\frac{1}{2}e^{-x}$

Figure 9



$y=\tanh x$

Figure 10

Derivatives of the Hyperbolic Functions

$$\frac{d}{dx}(\sinh x)=\cosh x \qquad \frac{d}{dx}(\coth x)=-\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\cosh x)=\sinh x \qquad \frac{d}{dx}(\operatorname{sech} x)=-\operatorname{sech} x\tanh x$$

$$\frac{d}{dx}(\tanh x)=\operatorname{sech}^2 x \qquad \frac{d}{dx}(\operatorname{csch} x)=-\operatorname{csch} x\coth x$$

Definition of the Inverse Hyperbolic Functions

$$\sinh^{-1}x=\ln\left(x+\sqrt{x^2+1}\right) \qquad (x\in R)$$

$$\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right) \qquad (x\ge 1)$$

$$\tanh^{-1}x=\frac{1}{2}\ln\frac{1+x}{1-x} \qquad (|x|<1)$$

$$\coth^{-1}x=\frac{1}{2}\ln\frac{x+1}{x-1} \qquad (|x|>1)$$

$$\operatorname{sech}^{-1}x=\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad (0<x<1)$$

$$\operatorname{csch} x=\ln\left(\frac{1}{x}+\frac{\sqrt{1-x^2}}{|x|}\right) \qquad (x\ne 0)$$

Derivatives of the Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}} \qquad (x\in R)$$

$$\frac{d}{dx}(\cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}} \qquad (x>1)$$

$$\frac{d}{dx}(\tanh^{-1}x)=\frac{1}{1-x^2} \qquad (|x|<1)$$

$$\frac{d}{dx}(\coth^{-1}x)=\frac{1}{1-x^2} \qquad (|x|>1)$$

1) http://www.math.ucdavis.edu/~kouba/CalcOneDIRECTORY/invtrigderivdirectory/InvTrigDeriv.html
2) http://en.wikipedia.org/wiki/Hyperbolic_function