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Easing into Eigenvectors and Eigenvalues in Introductory Linear Algebra

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Abstract

We provide a series of exercises for a first linear algebra course that introduces the concepts of eigenvalue, eigenvector, and diagonalization. Using the simple, geometric idea of fixed vectors and fixed directions, the basic aspects of eigenvectors as multiplicative invariants of a matrix are emphasized. These exercises can be assigned as soon as students have learned about matrix multiplication, linear combinations of vectors, linear independence, and the solution of small linear systems.
KEYWORDS: Eigenvalue, eigenvector, diagonalization, fixed vector, fixed direction, pedagogy.

1 INTRODUCTION

Many students find their initial encounters with eigenvalues and eigenvectors frustratingly difficult and uninformative. We believe that for such students, there are too many interrelated concepts involved in the standard formulation of the eigenproblem. These concepts include eigenvalues as roots of some determinantal polynomial, eigenvalues as variables in a matrix equation that also involves a vector of variables, and eigenvalues and eigenvectors as algebraic objects without geometric meaning.

Rather than be perceived as a unification of the seemingly disparate topics of matrix-vector multiplication, determinants, roots of polynomials, linear functions, solutions of linear systems, and vector spaces into one coherent whole, the introduction of eigenvalues and eigenvectors is perceived by many students as the joining of individually confusing ideas into an incomprehensible and overwhelming morass. Most current introductory linear algebra texts introduce eigenvalues first as the roots of the characteristic polynomial. For students, this involves much new terminology and several new concepts. The terminology includes characteristic polynomial and characteristic equation, a distinction that many students cannot make, and the key word, eigenvalue, whose germanic name provides no hint to its connection with the matrix or the determinantal polynomial. This introduction of eigenvalues is often the first appearance of the determinant as a function with variable input rather than as a number associated with a particular, usually numerical, matrix. Further, both the matrix and the parameter appear in a complicated fashion in the argument of this function: $\det(\lambda I_n - A)$. After students have struggled through the complicated mechanics of computing the characteristic polynomial, they are simultaneously confronted by the nightmare of factorization and with the apparently difficult conceptual relationship between the roots of a polynomial equation and the factors of the polynomial. Difficulty with even elementary factorization stymies student efforts to compute, let alone understand, eigenvalues. Unfortunately, there is little reason to expect an improvement in algebraic skills among future students. Indeed, as more students experience reformed calculus and reformed college algebra with their clear deemphasis on algebraic manipulation, skills in symbolic computation and factorization will further erode. Assuming that students survive the introduction of eigenvalues as the roots of the characteristic polynomial, they are then typically confronted by what seems to be an entirely different concept of eigenvalue, namely the eigenvalue as the unknown scalar in the matrix equation $Ax = \lambda x$. The theoretical web that connects the roots of the characteristic polynomial to the scalar in the matrix equation is, for many students, a sacramental mystery. This should not be a surprise when one considers the breadth of ideas involved: The presence of a vector x on *both* sides of a linear system of equations, the interdependence between the unknown scalar λ and the vector x of unknowns, the magical appearance of I_n in the re-arranged equation $(A - \lambda I_n)x = 0$, the relationship between nontrivial solutions x and the singularity of the matrix $A-\lambda I_n$, the relationship between the singularity of $A - \lambda I_n$ and the zero value of $\det(A - \lambda I_n)$, and finally, the equality of $\det(A - \lambda I_n)$ and $\det(\lambda I_n - A)$ when the former is zero. Absent from this entire development until possibly the very end is the *geometry* that underlies eigenvalues and eigenvectors.

We attempt to ease students into working with eigenvalues and eigenvectors by beginning with a geometric, matrix multiplication based approach that focuses initially on the eigenvectors rather than the eigenvalues. The initial concept in our approach is that of a *fixed vector,* that is, a vector v satisfying $Av = v$. After exploring fixed vectors, students are introduced in a subsequent assignment to the concept of a *fixed direction* for a matrix, that is, a vector v such that Av points in the same or opposite direction as v. In particular, students work with $Av = \lambda v$, where A is chosen so that λ and v are real, which has a very concrete geometric interpretation. Using the interpretation of Av as the linear combination of the columns of A weighted by the scalar entries of v, we can then lead students toward solving $AV = VM$ for M where V has independent columns and where M is (initially) a diagonal matrix. A later assignment guides students to examine issues of complex eigenvalues and eigenvectors, and of nondiagonalizability.

One key advantage of the approach discussed in this paper is that students can be introduced to eigenvalues, eigenvectors, and (non)diagonalizability very early in their first linear algebra course, and this introduction is graduated through a series of progressively more complicated assignments.

One natural question that the reader might ask is why we have studiously avoided using the words *eigenvector* and *eigenvalue*, and instead have introduced the nonstandard terminology of *fixed vectors* and *fixed directions*. The answer is in two parts. First, the nonstandard terminology explicitly captures what is special about these vectors. Second, the use of nonstandard terms prevents students from simply scanning the index of their textbook for the terms *eigenvector* and *eigenvalue*, and then rummaging through the relevant sections to find the textbook author's solutions to the assignment exercises. Knowledge gained by students through their own cogitation is likely to be longer retained, and perhaps even appreciated for its beauty.

Finally, we must comment on *what these assignments are not*: These assignments are *not* a replacement for traditional explanations of how one actually determines the eigenvalues of an arbitrary matrix. We make no claim whatsoever of that. Rather, it is our intention that these assignments be used early in a standard, one semester, matrix-oriented linear algebra course to foreshadow the later introduction of eigenvectors and eigenvalues, and by doing so, help students avoid the pitfalls that they encounter with the current approach. The structure of these assignments as a series of interlinked exercises is intended to make them function as short projects to be assigned as supplements to regular, text-based problem sets.

2 PREREQUISITES

The primary prerequisites for the exercises that follow are that students understand the concepts of a linear combination of vectors, matrix multiplication (including its interpretation in terms of linear combinations of columns), and linear independence of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. Certain problems require the ability to solve linear systems in two or three variables. A few questions concern bases for $\mathbb{R}^2$ or $\mathbb{R}^3$, but these can be posed in terms of spanning sets of independent vectors, or even simply in terms of finding a set of two or three independent vectors. With the exception of two questions, each occurring as a final exercise in a sequence of exercises, no mention is made of matrix inversion. An instructor who has already introduced matrix inverses, could choose to transform those questions implicitly pertaining to diagonalization into explicit questions about diagonalization. Conspicuously absent from the prerequisites is any knowledge of determinants or of matrix polynomials.

3 ASSIGNMENT ONE - FIXED VECTORS

We say that a square matrix A *fixes* a vector v if $Av = v$. The vector v is called a *fixed vector* for A.

1. Let $F = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

   (a) Show that each of the vectors $v_1 = [1, -1, 0, 0]^T$, $v_2 = [1, 0, -1, 0]^T$, and $v_3 = [1, 0, 0, -1]^T$ is fixed by F.

   (b) Is $v_4 = [1, 1, 1, 1]^T$ fixed by F?

(c) Is $5v_2$ fixed by F? Is $-v_3$ fixed by F? Is $5v_2 - v_3$ fixed by F?

Hint: In exercises 2 - 4, avoid looking at entries.

2. Show that the zero vector is fixed by every matrix.

3. Suppose that A fixes a vector v, and that c is a scalar. Does A fix cv?

4. Suppose that A fixes each of the vectors v and w. Does A fix $v + w$?

Hint: In exercises 5 - 7, look at entries. (Let $v = [a, b, c]^T$.)

5. Suppose that I_3 is the 3×3 identity matrix.
   (a) Does I_3 fix any nonzero vector v?
   (b) Choose a basis for $\mathbb{R}^3$.
   (c) Does I_3 fix each of the vectors in your basis?
   (d) Does I_3 fix each of the vectors in every basis for $\mathbb{R}^3$? Explain your answer.

6. Suppose that $D = diag(3, 2, 1)$.
   (a) Does D fix any nonzero vectors v?
   (b) How many independent vectors does D fix? Give a set of such vectors.
   (c) Does D fix a basis for $\mathbb{R}^3$? Explain your answer.

7. Suppose that $E = diag(1, 1, 0)$.
   (a) Does E fix any nonzero vectors v?
   (b) Does E fix a basis for $\mathbb{R}^3$? Explain your answer.

8. What properties must a 3×3 diagonal matrix have in order that it does fix a basis for $\mathbb{R}.^3$? (Hint: Write the product of a generic 3×3 diagonal matrix $D = diag(d_1, d_2, d_3)$ with a generic 3×1 vector v, and examine what being fixed means in terms of the entries of the product, the entries of the vector, and the entries of the matrix.)

9. Let $B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

   (a) Find all fixed vectors for B. (Hint: Set up and solve the linear equations corresponding to $Bv = v$ where $v \in \mathbb{R}^2$. Your variables will be the entries of v.)

   (b) How many independent fixed vectors does B have?

4 ASSIGNMENT TWO - FIXED VECTORS AND FIXED DIRECTIONS

Suppose that v is a nonzero vector. Recall that scalar multiples of v are vectors that point in the same (or opposite) direction as v, and further, if w is any vector that points in the same (or opposite) direction as v, then w must be a scalar multiple of v, that is, $w = cv$ for some scalar c.
We say that a square matrix A *fixes a direction* if there is a *nonzero* vector v for which Av has the same (or opposite) direction as v. That is, A fixes the direction specified by the nonzero vector v provided there exists a scalar λ such that $Av = \lambda v$. We say that the nonzero vector v *corresponds to a fixed direction* when $Av = \lambda v$ for some scalar λ.

1. Let $F = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

   (a) Show that each of the vectors $v_1 = [1, -1, 0, 0]^T$, $v_2 = [1, 0, -1, 0]^T$, and $v_3 = [1, 0, 0, -1]^T$ corresponds to a fixed direction for F for $\lambda = 1$.

   (b) Does $v_4 = [1, 1, 1, 1]^T$ correspond to a fixed direction for F for some value of λ?

   (c) Do $5v_2$ and $-v_3$ correspond to fixed directions for F for some value of λ?

   (d) Does $5v_2 - v_3$ correspond to a fixed direction for F for some value of λ?

(e) Does $5v_2 - v_4$ correspond to a fixed direction for F for some value of λ?

2. Suppose that A fixes the direction of v, and that c is a scalar. Does A fix the direction of cv?

3. Suppose that λ is positive. Can you give a description of what happens geometrically to vectors pointing in a fixed direction when they are multiplied by the matrix A? What if λ is negative? What if λ is zero?

4. Recall that a square matrix A *fixes* a vector v if $Av = v$.

   (a) Explain why saying that A fixes the vector v also means that A fixes the direction corresponding to v.

   (b) If A fixes the *direction* of some nonzero vector v, must it also fix the vector v?

   (c) Explain why all fixed directions for the matrix I_3 actually correspond to fixed vectors.

5. Let D be the matrix $D = diag(3, 2, 1)$.

   (a) Does D have any fixed directions? What values of λ occur? (Hint: Let $v = [a, b, c]^T$.)

   (b) Can you find a basis for $\mathbb{R}^3$ that consists entirely of vectors that correspond to fixed directions of D? Explain your answer.

6. Based on your experience with the matrices F and D, if A fixes the direction given by a nonzero vector v, and if A fixes the direction given by a nonzero vector w, must A fix the direction given by the vector $v + w$? If you need to impose any additional conditions so that A fixes the direction of $v + w$ when A fixes the directions of v and w, what are they?

7. Let E be the matrix $E = diag(1, 1, 0)$.

   (a) Find all fixed directions for E. What values of λ occur? (Hint: Let $v = [a, b, c]^T$.)

   (b) Can you find a basis for $\mathbb{R}^3$ that consists entirely of vectors that correspond to fixed directions? Explain your answer.

8. Let S be the 4×4 matrix whose columns are the vectors v_1, v_2, v_3 and v_4 given in problem 1 above.

   (a) Verify that the columns of S are independent.

   (b) Describe each column of FS in terms of linear combinations of the columns of S.

   (c) Find a matrix M such that $FS = SM$. (Hint: Use the previous part.)

9. Let $G = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$.

   (a) Find two independent, fixed directions for G. What scalars λ did you use? (Hint: Let $v = [a, b]^T$, and solve the system of equations corresponding to $Gv = \lambda v$.)

   (b) Call the direction vectors you found in the previous part w_1 and w_2. Let T be the matrix whose columns are your two vectors. Describe each column of GT in terms of linear combinations of the columns of T.

   (c) Find a matrix N such that $GT = TN$. (Hint: Use the previous part.)

   (d) Find the inverse matrix T^{-1}, and compute $T^{-1}GT$.

5 ASSIGNMENT THREE - MORE ON FIXED DIRECTIONS

Recall that we say that a square matrix A *fixes a direction* if there is a nonzero vector v for which Av has the same (or opposite) direction as v. That is, A fixes the direction specified by the nonzero vector v provided there exists a scalar λ such that $Av = \lambda v$.

1. Let $A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$.

   (a) Find a vector v_1 corresponding to a fixed direction for A with $\lambda = 3$.

(b) Find a vector v_2 corresponding to a fixed direction for A with $\lambda = 1$.

(c) Let S be the 2×2 matrix whose columns are v_1 and v_2 . Verify that the columns of S are independent.

(d) Describe each column of AS in terms of linear combinations of the columns of S.

(e) Find a matrix H such that $AS = SH$.

(f) Compute A^2S and SH^2. How are these two products related? Which was easier to compute?

(g) Compute A^6S and SH^6. How are these two products related? Which was easier to compute?

(h) Suppose you needed to compute $A^{100}S$, how might you compute it easily? Justify your answer.

2. Suppose that M, N and P are square matrices satisfying $MP = PN$. Find a simple relationship between M^2P and PN^2. Using your argument, can you find and explain a relationship between M^kP and PN^k where k is a positive integer?

3. Suppose that M, N and P are square matrices, that $MP = PN$, and that P has independent columns. Notice that since P is square and has independent columns, it must be invertible. In light of the previous exercise, find a simple expression for the powers of M in terms of powers of N and powers of P. If N is a diagonal matrix, what does this suggest about computing powers of M?

4. Let $B = \begin{bmatrix} 3 & -1 \\ 0 & 3 \end{bmatrix}$.

(a) Find a vector w_1 corresponding to a fixed direction for B with $\lambda = 3$.

(b) Show that B has no other fixed directions independent from the direction given by w_1.

(c) Let T be the 2×2 matrix whose first column is w_1 and whose second column is $[x, y]^T$. What condition(s) on x and y are necessary in order that the columns of T be independent?

(d) Can you find values for the scalars d, x and y, with $y \neq 0$, such that $BT = TD$ where $D = diag(3, d)$?

5. Let $C = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$.

   (a) Can you find any real vectors corresponding to fixed directions for C?

   (b) If you allow *complex vectors* and *complex scalars,* show that $u_1 = [1, i]^T$ is a fixed direction for C. What is the required complex scalar λ?

   (c) Find a second fixed direction for C. (Hint: Look for a vector u_2 of the form $u_2 = [1, *]^T$ where $*$ is a complex number other than i.) What is the required complex scalar λ?

   (d) Let U be the 2×2 matrix whose columns are the vectors u_1 and u_2. Show that the columns of U are independent.

   (e) Describe the columns of CU in terms of linear combinations of the columns of U.

   (f) Find a matrix K such that $CU = UK$.

6 CONCLUDING REMARKS

Somehow one suspects that these exercises are old wine in new bottles. Perhaps a generation ago, linear algebra was primarily an advanced topic reserved for mathematics majors, rather than the matrix-oriented service course for client disciplines that it has increasingly become. In that former period, linear algebra texts apparently focused more on transformations of abstract vectors in an abstract vector space, and rather less on the matrix representations of those transformations multiplying vectors in $\mathbb{R}^n$. In the context of transformations, the idea of an eigenvector as an invariant of the transformation is much clearer than it is when viewed in the algebraic context as the solution to the linear system $(A - \lambda I)v = 0$.

Although we have referred to the geometric aspect of eigenvectors, we have used only the simplest geometric aspect: invariance of direction. Some authors of textbooks have invested much more time and detail into the geometric aspects of eigenvectors in the case of 2×2 matrices and vectors in $\mathbb{R}^2$, classifying matrices based on their geometric transformation of the plane.(See [**?**], [**?**].)

Finally, we would like to suggest that these exercises and their prerequisites are very much in the spirit of the recommendations of the Linear Algebra Curriculum Study Group [**?**].

7 REFERENCES

References

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