

# Teaching Linear Algebra at University

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## Abstract

Linear algebra represents, with calculus, the two main mathematical subjects taught in science universities. However this teaching has always been difficult. In the last two decades, it became an active area for research works in mathematics education in several countries. Our goal is to give a synthetic overview of the main results of these works focusing on the most recent developments. The main issues we will address concern:

- the epistemological specificity of linear algebra and the interaction with research in history of mathematics
- the cognitive flexibility at stake in learning linear algebra
- three principles for the teaching of linear algebra as postulated by G. Harel
- the relation between geometry and linear algebra
- an original teaching design experimented by M. Rogalski

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## 1. Introduction

In most countries, science-orientated curricula in the first two years at university consist of courses in two main subjects, namely, calculus and linear algebra. The difficulties in these two fields are of different nature. Mathematics education research first developed works on calculus, but in the past 20 years, an increasing number of studies has been carried out about the teaching of linear algebra. One can distinguish roughly two main traditions in the teaching of linear algebra: one focuses on the study of formal vector spaces while the other proposes a more analytical approach based on the study of  $\mathbb{R}^n$  and matrix calculus. Between these two orientations, there exist a continuum of teaching designs, in which each pole is more or less dominant. However, the teaching of linear algebra is universally

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recognised as difficult. Students usually feel that they land on another planet, they are overwhelmed by the number of new definitions and the lack of connection with previous knowledge. On the other hand, teachers often feel frustrated and disarmed when faced with the inability of their students to cope with ideas that they consider to be so simple. Usually, they incriminate the lack of practice in basic logic and set theory or the impossibility for the students to use geometrical intuition. These complaints have a certain validity, but the few attempts at remedying this state of affairs - with the teaching of Cartesian geometry or/and logic and set theory prior to the linear algebra course - did not seem to improve the situation substantially. The aim of this text is to give an account of the main trends in this area of mathematics education research.

## 2. Historical analyses

An epistemological analysis of the history of linear algebra is a way to reveal some possible sources of students' difficulties as well an inspiration in the design of activities for students. Several works have been carried out in this direction (see [4], [6], [8], [17] and [24]). In this paper, we will give an account of only one of the main result of this type of research. It concerns the last phase of the genesis of the theory of vector spaces, whose roots can be found in the late nineteenth century, but really started only after 1930. It corresponds to the axiomatisation of linear algebra, that is to say a theoretical reconstruction of the methods of solving linear problems, using the concepts and tools of a new axiomatic central theory. These methods were operational but they were not explicitly theorised or unified. It is important to realise that this axiomatisation did not, in itself, allow mathematicians to solve new problems; rather, it gave them a more universal approach and language to be used in a variety of contexts (functional analysis, quadratic forms, arithmetic, geometry, etc.). The axiomatic approach was not an absolute necessity, except for problems in non-denumerable infinite dimension, but it became a universal way of thinking and organising linear algebra. Therefore, the success of axiomatisation did not come from the possibility of reaching a solution to unsolved mathematical problems, but from its power of generalisation and unification and, consequently, of simplification in the search for methods for solving problems in mathematics.

As a consequence, one of the most noticeable difficulties encountered in the learning of unifying and generalising concepts are associated with the pre-existing, related elements of knowledge or competencies of lower level. Indeed, these need to be integrated within a process of abstraction, which means that they have to be looked at critically, and their common characteristics have to be identified, and then generalised and unified. From a didactic point of view, the difficulty is that any linear problem within the reach of a first year university student can be solved without using the axiomatic theory. The gain in terms of unification, generalisation and simplification brought by the use of the formal theory is only visible to the expert.

One solution would be to give up teaching the formal theory of vector spaces. However, many people find it important that students starting university mathe-

matics and science studies get some idea about the axiomatic algebraic structures of which vector space is one of the most fundamental. In order to reach this goal, the question of formalism cannot be avoided. Therefore, students have to be introduced to a certain type of reflection on the use of their previous elements of knowledge and competencies in relation with new formal concepts. This led Dorier, Robert, Robinet and Rogalski to introduce what they called ‘*meta level activities*’ (see [5], [6], [9], [11], [21] [22] and [23]). These activities are introduced and maintained by an explicit discourse on the part of the teacher about the significance of the introduced concepts for the general theory, their generalising and unifying character, the change of point of view or a theoretical detour that they offer, the types of general methods they lead to, etc. It hinges on the general attitude of the teacher who induces a constant underlying meta-questioning concerning new possibilities or conceptual gains provided by the use of linear algebra concepts, tools and methods.

### 3. Cognitive flexibility

One of the main difficulties in learning linear algebra has to do with the variety of languages, semiotic registers of representation, points of view and settings through which the objects of linear algebra can be represented. Students have to distinguish these various ways of representing objects of linear algebra, but they also need to translate from one to another type and, yet, not confuse the objects with their different representations. These abilities could be referred by the general notion of cognitive flexibility. This question is central in several works on the teaching and learning of linear algebra.

Students’ difficulties with the formal aspect of the theory of vector spaces are not just a general problem with formalism but mostly a difficulty of understanding the specific use of formalism within the theory of vector spaces and the interpretation of the formal concepts in relation with more intuitive contexts like geometry or systems of linear equations, in which they historically emerged. Various diagnostic studies conducted by Dorier, Robert, Robinet and Rogalski pointed to a single massive obstacle appearing for all successive generations of students and for nearly all modes of teaching, namely, what these authors termed *the obstacle of formalism* (see [6], [7], [10] and [27]).

In [16], Hillel distinguished three basic languages used in linear algebra: the ‘abstract language’ of the general abstract theory, the ‘algebraic language’ of the  $\mathbb{R}^n$  theory and the ‘geometric language’ of the two- and three-dimensional spaces. The ‘opaqueness’ of the representations seems to be ignored by lecturers, who constantly shift the notations and modes of description, without alerting the students in any explicit way. By far, the most confusing case for students is the shift from the abstract to the algebraic representation when the underlying vector space is  $\mathbb{R}^n$ . In this case, an  $n$ -tuple (or a matrix) is represented as another  $n$ -tuple (or matrix) relative to another basis. This confusion leads to persistent mistakes in students’ solutions related to reading the values of a linear transformation given by a matrix in a basis (see [15]). Parallel with the three languages identified by Hillel, Sierpinska et al. distinguish three modes of thinking that have led to the development of these

languages and are necessary for an understanding of the domain: the ‘synthetic-geometric’, ‘analytic-arithmetic’, and ‘analytic-structural’ (see [29]).

In [12], Duval defined *semiotic representations* as productions made by the use of signs belonging to a system of representation which has its own constraints of meaning and functioning. Semiotic representations are, according to him, absolutely necessary in mathematical activity, because its objects cannot be directly perceived and must, therefore, be represented. Moreover, semiotic representations play an essential role in developing mental representations, in accomplishing different cognitive functions (objectification, calculation, etc.), as well as in producing knowledge.

In her work, Pavlopoulou (see [8] pp. 247-252) applied and tested Duval’s theory in the context of linear algebra. She distinguished between three registers of semiotic representation of vectors: the graphical register (arrows), the table register (columns of coordinates), and the symbolic register (axiomatic theory of vector spaces). Through several studies, she has shown that the question of registers, especially as regards conversion, is not usually taken into account either in teaching or in textbooks. She also identified a number of student’s mistakes that could be interpreted as a confusion between an object and its representation (especially a vector and its geometrical representation) or as a difficulty in converting from one register to another.

The research of Alves-Dias (see [1] and [8] pp. 252-256), an extension of Pavlopoulou’s, generalised the necessity of conversions from one semiotic register to another for the understanding of linear algebra to the necessity of ‘cognitive flexibility’. Moreover, on the basis of Rogalski’s previous work (see [25], [26], [27] and [28]), she focused her study on the question of articulation between the Cartesian and parametric representations of vector subspaces, which is not a mere question of change of register, but deals with more complex cognitive processes involving the use of concepts like rank and duality. Indeed, when a subspace  $V$  is represented by Cartesian equations, finding a parametric representation of  $V$  mostly consists in finding a set of generators of  $V$ , which is not just a change of register, nor an elementary cognitive process, even if it is much easier when the dimension ‘ $d$ ’ of  $V$  is known. In any case, competencies with regard to the concept of rank and duality are indispensable. Moreover, in order to avoid easy mistakes in calculations or reasoning, it is necessary to be able to have some control over the results obtained. Alves Dias showed that in textbooks and classes, in general, the tasks offered to students are very limited in terms of flexibility. She developed a series of exercises that required the student to mobilise more changes of settings or registers and to exert explicit control via the concepts of rank and duality. Her experimentations demonstrated a variety of difficulties for the students. For instance, students often identified one type of representation exclusively through semiotic characteristics (a representation with  $x$ ’s and  $y$ ’s would be considered as obviously Cartesian) without questioning the meaning of the representation. Concerning the means of control over the validity of the statements by the students and anticipation of results or answers to problems, she found that a theorem like:  $\dim E = \dim \text{Ker } f + \dim \text{Im } f$ , is known and used correctly by many students, but it is very seldom used for those

purposes even in cases in which it would immediately bring up a contradiction with the result obtained, or in cases in which it offered valuable information in order to anticipate the correct answer.

In [15], Hillel and Sierpinska stressed that a linear algebra course which is theoretically rather than computationally framed requires a level of thinking that is based on what has been termed by Piaget and Garcia as the ‘trans-object level of analysis’ which consists in the building of conceptual structures out of what, at previous levels, were individual objects, actions on these objects, and transformations of both the objects and actions (see [18] p. 28). A similar claim was made by Harel in [14], in his assertions that a substantial range of mental processes must be encapsulated into conceptual objects by the time students get to study linear algebra. The difficulty of thinking at the trans-object level leads some students to develop ‘defense mechanisms’ (to ‘survive’ the course), consisting in trying to produce a written discourse formally similar to that of the textbook or of the lecture but without grasping the meaning of the symbols and the terminology. This appeared as a major problem for Sierpinska, Dreyfus and Hillel, and the team set out to design an entry into linear algebra that would make this behaviour or attitude less likely to appear in students (see [30]). The designed teaching-learning situations were set in a dynamic geometry environment (Cabri-geometry II) extended by several macro-constructions for the purposes of representing a two-dimensional vector space and its transformations (see [31] and [32]). Further analysis of the students’ behaviour in the experimented situations led Sierpinska to postulate certain features of their thinking that could be held partly responsible for their erroneous understandings and difficulties in dealing with certain problems (especially the problem of extending a transformation of a basis to a linear transformation of the whole plane). She proposed that these features be termed “a tendency to think in ‘practical’ rather than ‘theoretical’ ways” (see [32]). The distinction between these two ways of thinking was inspired by the Vygotskian notion of scientific, as opposed to spontaneous or everyday concepts. The behaviour of students who were encountering difficulties in the experimentations suggested that their ways of thinking had the features of practical thinking rather than theoretical thinking. In particular they had trouble going beyond the appearance of the graphical and dynamic representations in Cabri that they were observing and manipulating: their relation to these representations was ‘phenomenological’ rather than ‘analytic’. By far the most blatant feature of the students’ practical thinking was their tendency to base their understanding of an abstract concept on ‘prototypical examples’ rather than on its definition. For example, linear transformations were understood as ‘rotations, dilations, shears and combinations of these’. This way of understanding made it very difficult for them to see how a linear transformation could be determined by its value on a basis, and consequently, their notion of the matrix of a linear transformation remained at the level of procedure only.

#### **4. Three principles for the teaching of linear algebra**

In [14], Harel posits three ‘principles’ for the teaching of linear algebra, inspired by Piaget’s psychological theory of concept development: the *Concreteness Principle*, the *Necessity Principle* and the *Generalisability Principle*.

The Concreteness Principle states, “For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student’s eyes; that is to say, the student has mental procedures that can take these objects as inputs”. This principle is violated whenever the general concept of vector space is taught as a generalisation from less abstract structures, to students who have not (yet) constructed the elements of these structures as mental entities on which other mental operations can be performed. Starting from the premise that students build their understanding of a concept in a context that is concrete to them, Harel conclude that a sustained emphasis on a geometric embodiment of abstract linear algebra concepts produce a quite solid basis for students’ understanding. He insisted, however, that it would be incorrect to conclude that a linear algebra course should start with geometry and build the algebraic concepts through some kind of generalisation from geometry. A teaching experiment built on this premise allowed Harel to observe that when geometry is introduced before the algebraic concepts have been formed, many students remain in the restricted world of geometric vectors, and do not move up to the general case.

The Necessity Principle — For students to learn, they must see an (intellectual, as opposed to social or economic) need for what they are intended to be taught — is based on the Piagetian assumption (which has also been adopted by the Theory of Didactic Situations elaborated by Brousseau in [2]) that knowledge develops as a solution to a problem. If the teacher solves the problems for the students and only asks them to reproduce the solutions, they will learn how to reproduce teacher’s solutions, not how to solve problems. Deriving the definition of vector space from a presentation of the properties of  $\mathbb{R}^n$  is an example of a violation of the necessity principle.

The last, Generalisability Principle postulated by Harel, is concerned more with didactic decisions regarding the choice of teaching material than with the process of learning itself. “When instruction is concerned with a ‘concrete’ model, that is a model that satisfies the Concreteness Principle, the instructional activities within this model should allow and encourage the generalisability of concepts.” This principle would be violated if the models used for the sake of concretisation were so specific as to have little in common with the general concepts they were aimed at. For example, the notion of linear dependence introduced in a geometric context defined through collinearity or co-planarity is not easily generalisable to abstract vector spaces. Harel’s work inspired curriculum reform in the US (see [3]), as well as textbook authors (see [33]).

## 5. Geometry and linear algebra

In [19], Robert, Robinet and Tenaud designed and experimented with a geometric entry into linear algebra. The aim was to overcome the obstacle of formalism by giving a more ‘concrete’ meaning to linear algebra concepts, in particular,

through geometrical figures that could be used as metaphors for general linear situations in more elaborate vector spaces. However, as Harel noticed after them in his study mentioned above, the connection with geometry proved to be problematic. Firstly, geometry is limited to three dimensions and therefore some concepts, like rank, for instance, or even linear dependence, have a quite limited field of representation in the geometric context. Moreover, it is not rare that students refer to affine subspaces instead of vector subspaces when working on geometrical examples within linear algebra.

In her work, Gueudet-Chartier (see [13] and [8] pp. 262-264) conducted an epistemological study of the connection between geometry and linear algebra, using the evidence from both historical and modern texts. She found that the necessity of geometric intuition was very often postulated by textbooks or teachers of linear algebra. However, in reality, the use of geometry was most often very superficial. Moreover, some students would use geometrical representations or references in linear algebra, without this always being to their advantage. Indeed, some of them could not distinguish the affine space from the vector space structure; they also often could not imagine a linear transformation that would not be a geometric transformation. In other words, the geometrical reference acted as an obstacle to the understanding of general linear algebra. On the other hand, some very good students were found to use geometric references very rarely. They could operate on the formal level without using geometrical representations. It seems that the use of geometrical representations or language is very likely to be a positive factor, but it has to be controlled and used in a context where the connection is made explicit.

## 6. An original teaching experiment

Most of the research conducted in France on the teaching and learning of linear algebra has been more or less directly connected with an experimental course implemented by Rogalski (see [9], [25], [26], [27] and [28]). This course was built on several interwoven and long-term strategies, using meta level activities as well as changes of settings (including intra-mathematical changes of settings), changes of registers and points of view, in order to obtain a substantial improvement in a sufficient number of students. ‘Long-term strategy’ (see [20]) refers to a type of teaching that cannot be divided into separate and independent modules. The long-term aspect is vital because the mathematical preparation and the changes in the ‘didactic contract’ (see [2]) have to operate over a period which is long enough to be efficient for the students, in particular as regards assessment. Moreover, the long-term strategy refers to the necessity of taking into account the non-linearity of the teaching due to the use of change in points of view, implying that a subject is (re)visited several times in the course of the year.

Rogalski’s teaching design has the following main characteristics:

- In order to take into account the specific epistemological nature of the concepts, some activities are introduced, at a favourable and precise time of the teaching, in order to induce a reflection on a ‘meta’ level.
- A fairly long preliminary phase precedes the actual teaching of elementary

concepts of linear algebra. It prepares the students to understand, through ‘meta’ activities, the unifying role of these concepts.

- As much as possible, changes of settings and points of view are used explicitly and are discussed.
- Finally, the concept of rank is given a central position in this teaching.

For a long-term teaching design, it is difficult to choose the time suitable for its evaluation, as interference may occur due to students’ own organisation of their time and work, in a way that cannot be kept under control. Thus, phenomena of maturing, depending on students’ level of involvement (which varies during the year) are difficult to take into account in the evaluation of the teaching. Moreover, such a global teaching design cannot be evaluated by usual comparative analyses, because the differences with the standard course are too important. However, internal evaluations have been conducted, showing several positive effects, even if some questions remain open.

## 7. Conclusions

Mathematics education research cannot give a miraculous solution to overcome all the difficulties in learning and teaching linear algebra. Various works have consisted in diagnoses of students’ difficulties, epistemological analyses and experimental teaching, offering local remediation. Nevertheless, these works lead to new questions, problems and difficulties. Yet, this should not be interpreted as a failure. Improving the teaching and learning of mathematics cannot consist in one remediation valid for all. cognitive processes and mathematics are far too complex for such an idealistic simplistic view. It is a deeper knowledge of the nature of the concepts, and the cognitive difficulties they enclose, that helps teachers make their teaching richer and more expert; not in a rigid and dogmatic way, but with flexibility. In this sense, in several countries, mathematics education research has influenced curriculum reforms, in an non-formal way, or sometimes very officially, like in North-America through the Linear Algebra Curriculum Study Group (see [3]).

## References

- [1] M. Alves Dias & M. Artigue, Articulation Problems between Different Systems of Symbolic Representations in Linear Algebra, in *The Proceedings of PME19*, Universidade Federal de Pernambuco, Recife, Brazil, 1995, Volume 2, 34–41.
- [2] G. Brousseau, *Theory of Didactic Situations in Mathematics*, Kluwer Academic Publishers, Dordrecht, 1997.
- [3] D. Carlson, C. Johnson, D. Lay, & A. Porter, The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra, *College Mathematics Journal*, 24 (1993), 41–46.
- [4] J.-L. Dorier, A General Outline of the Genesis of Vector Space Theory, *Historia Mathematica*, 22.3 (1995), 227–261.

- [5] J.-L. Dorier, Meta Level in the Teaching of Unifying and Generalizing Concepts in Mathematics, *Educational Studies in Mathematics*, 29.2 (1995), 175–197.
- [6] J.-L. Dorier (Ed.), *L'Enseignement de l'Algèbre Linéaire en Question*, La Pensée Sauvage éditions, Grenoble, 1997.
- [7] J.-L. Dorier, The Role of Formalism in the Teaching of the Theory of Vector Spaces. *Linear Algebra and its Applications* (275), 1.4 (1998), 141–160.
- [8] J.-L. Dorier, *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000.
- [9] J.-L. Dorier, A. Robert, J. Robinet & M. Rogalski, On a Research Program about the Teaching and Learning of Linear Algebra in First Year of French Science University, *International Journal of Mathematical Education in Sciences and Technology*, 31.1 (2000), 27–35.
- [10] J.-L. Dorier, A. Robert, J. Robinet & M. Rogalski, The Obstacle of Formalism in Linear Algebra, in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 85–94.
- [11] J.-L. Dorier, A. Robert, J. Robinet & M. Rogalski, The Meta Lever, in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 151–176.
- [12] R. Duval, *Sémiosis et Pensée Humaine. Registres Sémiotiques et Apprentissages Intellectuels*, Peter Lang, Bern, 1995.
- [13] G. Gueudet-Chartier, *Rôle du Géométrique dans l'Enseignement et l'Apprentissage de l'Algèbre Linéaire*, Thèse de Doctorat de l'Université Joseph Fourier, Laboratoire Leibniz, Grenoble, 2000.
- [14] G. Harel, Principles of Learning and Teaching Mathematics, With Particular Reference to the Learning and Teaching of Linear Algebra: Old and New Observations, in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 177–189.
- [15] J. Hillel & A. Sierpiska, On One Persistent Mistake in Linear Algebra, in *The Proceedings PME 18*, University of Lisbon, Portugal, 1994, 65–72.
- [16] J. Hillel. Modes of Description and the Problem of Representation in Linear Algebra. in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 191–207.
- [17] G. Moore, The Axiomatization of Linear Algebra, *Historia Mathematica*, 22.3, 1995, 262–303.
- [18] J. Piaget & R. Garcia, *Psychogenesis and the History of Science*, Columbia University Press, New York, 1983.
- [19] A. Robert, J. Robinet & I. Tenaud, *De la Géométrie à l'Algèbre Linéaire* - Brochure 72, IREM de Paris VII, 1987.
- [20] A. Robert, Projets Longs et Ingénierie pour l'Enseignement Universitaire: Questions de Problématique et de Méthodologie. Un exemple: un Enseignement Annuel de Licence en Formation Continue, *Recherches en Didactique des Mathématiques*, 12.2/3 (1992), 181–220.
- [21] A. Robert, & J. Robinet, Prise en Compte du Méta en Didactique des Mathématiques, *Recherches en Didactique des Mathématiques*, 16.2 (1996), 145–176.

- [22] A. Robert, Outils d'Analyse des Contenus Mathématiques Enseignés au Lycée et l'Université, *Recherches en Didactique des Mathématiques*, 18.2 (1998), 191–230.
- [23] A. Robert, Level of Conceptualization and Secondary School Mathematics Education, in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 125–131.
- [24] J. Robinet, *Esquisse d'une Genèse des Concepts d'Algèbre Linéaire - Cahier de Didactique des Mathématiques n29*, IREM de Paris VII, 1986.
- [25] M. Rogalski, *Un Enseignement de l'Algèbre Linéaire en DEUG A Première Année - Cahier de Didactique des Mathématiques n53*, IREM de Paris 7, 1991.
- [26] M. Rogalski. L'Enseignement de l'Algèbre Linéaire en Première Année de DEUG A, *La Gazette des Mathématiciens*, 60 (1994), 39–62.
- [27] M. Rogalski, Teaching Linear Algebra: Role and Nature of Knowledge in Logic and Set Theory which Deal with Some Linear Problems, in *The Proceedings PME 20*, Valencia Universidad., Spain, 1996, Volume 4, 211–218.
- [28] M. Rogalski, The Teaching Experimented in Lille, in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 133–149.
- [29] A. Sierpinska, A. Defence, T. Khatcherian & L. Saldanha, A propos de trois modes de raisonnement en algèbre linéaire, in J.-L. Dorier (Ed.), *L'Enseignement de l'Algèbre Linéaire en Question*, La Pensée Sauvage éditions, Grenoble, 1997, 249–268.
- [30] A. Sierpinska, T. Dreyfus, & J. Hillel, Evaluation of a Teaching Design in Linear Algebra: The Case of Linear Transformations, *Recherches en Didactique des Mathématiques*, 19.1 (1999), 7–40.
- [31] A. Sierpinska, J. Trgalova, J. Hillel, & T. Dreyfus, Teaching and Learning Linear Algebra with Cabri. Research Forum paper, in *The Proceedings of PME 23*, Haifa University, Israel, 1999, Volume 1, 119–134.
- [32] A. Sierpinska, On Some Aspects of Students' Thinking in Linear Algebra. in J.-L. Dorier (Ed.), *On the Teaching of Linear Algebra*, Kluwer Academic Publishers, Dordrecht, 2000, 209–246.
- [33] F. Uhlig, *Transform Linear Algebra*, Prentice-Hall, Upper Saddle River, 2002.