The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra

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There is a growing concern that the linear algebra curriculum at many schools does not adequately address the needs of the students it attempts to serve.

In recent years, demand for linear algebra training has risen in client disciplines such as engineering, computer science, operations research, economics, and statistics. At the same time, hardware and software improvements in computer science have raised the power of linear algebra to solve problems that are orders of magnitude greater than dreamed possible a few decades ago. Yet in many courses, the importance of linear algebra in applied fields is not communicated to students, and the influence of the computer is not felt in the classroom, in the selection of topics covered or in the mode of presentation. Furthermore, an overemphasis on abstraction may overwhelm beginning students to the point where they leave the course with little understanding or mastery of the basic concepts they may need in later courses and their careers.

In response to these problems, we formed the Linear Algebra Curriculum Study Group in January 1990. Our goal has been to initiate substantial and sustained national interest in improving the undergraduate linear algebra curriculum. Funded by an NSF grant, the Study Group was enlarged in August 1990 for a five-day workshop on the undergraduate linear algebra curriculum, held at the College of William and Mary. The workshop panel was broadly based, with a variety of interests ranging from pure to numerical linear algebra. Participants came from mathematics departments in different regions of the country, from public and private schools, from two-year and four-year colleges, and from universities having graduate programs in mathematics and engineering schools. In addition, consultants from client disciplines participated for one to three days, describing the role of linear algebra in their disciplines and suggesting ways in which the curriculum could be improved. The following five recommendations were generated at that workshop. They are presented here to stimulate further discussion and creative work on linear algebra in the mathematics curriculum.

Recommendations

1. The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines.

A first course should be recognized as a service course for a wide variety of disciplines. Some of the major client disciplines are: computer science, electrical engineering, other engineering fields such as aerospace engineering and systems engineering, physics, economics, statistics, and operations research. In addition, students interested in graduate work in mathematics form a small but important group whose needs must be considered.

Since most students currently take only one course in linear algebra, it is imperative that the course syllabus contain the topics and concepts needed most by the majority of the students. Excursions into generalizations and other issues should be made only if time permits. However, it should be possible to design a syllabus that includes both the essential topics and other concepts that either serve to emphasize the position of linear algebra within mathematics, reflect the unique mission of an institution, or respond to special interests and needs of the students.

Some applications of linear algebra should be included to give an indication of the pervasive use of linear algebra in many client disciplines. Such applications necessarily will be limited by the need to minimize technical jargon and information from outside the course. Students should see the course as one of the most potentially useful mathematics courses they will take as an undergraduate.

The level and mode of presentation must take into account the backgrounds and abilities of the students and also the way in which they are likely to use linear algebra later in their careers. Representatives from client disciplines have stressed the need for a solid and intellectually challenging course, with careful definitions and statements of theorems, and proofs that show relationships between various concepts and enhance understanding.

More study and thought must be given to identifying those topics that are essential for a first course. More contacts with professionals in other client disciplines should be made. Individual faculty should discuss their linear algebra course with colleagues in other departments, with the objective of making the course more valuable for their students. The Study Group has begun this dialog, meeting with distinguished representatives from several client disciplines. Though each discipline had its own particular needs, there was a general agreement about certain desirable features of a first course. These are discussed in the next recommendation.

2. Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course.

For many departments this may be a shift in focus rather than a significant change in content.

In the last quarter century the importance and usefulness of linear algebra has continued to grow and it is now an essential tool for industrial scientists. We believe that a first course in linear algebra should be taught in a way that reflects its new role as a scientific tool. This implies less emphasis on abstraction and more emphasis on problem solving and motivating applications. It does not imply less emphasis on rigor or theorem proving, but rather a change of focus from an abstract, inward-looking course to a more practical matrix-oriented course that meets the needs of not only mathematics students but also the students of the various client disciplines. The first course in linear algebra can and should be one of the most useful mathematics courses taken by college mathematics students.

A matrix-oriented linear algebra course should proceed from concrete, and in many cases practical, examples to the development of general concepts, principles, and the concomitant theory that simplifies and clarifies and makes linear algebra so powerful and useful.

Below we recommend a core syllabus for a first course in linear algebra and a collection of additional topics, some of which can be chosen as time permits. Since institutional needs and faculty opinions differ widely, agreement on a core content is the most one could hope to achieve. However, such agreement would make it easier to design subsequent courses in both mathematics departments and the departments of client disciplines. In addition, agreement on core topics would assist in the transfer of credits from two-year schools and between four-year schools.

The core syllabus contains material that we believe can be covered in 26–28 fifty-minute class periods. The remaining time can be used to include supplementary topics (some suggestions are listed below) that will help the course meet local needs or that the individual instructor feels are important. Part of the remaining time might also be used to cover the core syllabus more deeply. The syllabus should be taught with an awareness of the importance of technology in modern applications of linear algebra.

We hope that the proposed syllabus will serve as a catalyst for future discussion on the linear algebra curriculum and will eventually lead to widespread agreement on what the core content of a first course should be as we approach the 21st century. We invite colleagues to convey to us their ideas regarding the proposed syllabus, to try it as part of their first course, and to report their results to the mathematical community.

Core Syllabus

The prerequisite for this proposed course is the mathematical maturity associated with the successful completion of two semesters of calculus. The goal of the course should be mastery of these core topics as well as increasing problem-solving capability.

I. Matrix Addition and Multiplication — 3 days

This includes the normal topics of matrix addition, scalar multiplication, matrix multiplication, transposition, and their algebraic properties such as associativity of matrix multiplication. Operations with partitioned matrices. Motivate matrix multiplication and carefully examine three views of the product AB:

1. Ax is a linear combination of the columns of A, with coefficients from x; each column of AB is obtained by multiplying A by the corresponding column of B. Thus, each column of AB is a linear combination of the columns of A, with coefficients from the corresponding column of B. If D is a diagonal matrix, then AD is a scaling of the columns of A. If P is a permutation matrix, then AP is a permutation of the columns of A.
2. Similarly, the rows of AB are linear combinations of the rows of B.
3. AB is a sum of outer products (i.e., rank 1 matrices): $AB = \text{col}_1(A)\text{row}_1(B) + \cdots + \text{col}_k(A)\text{row}_k(B)$, when A is m by k and B is k by n.

II. Systems of Linear Equations — 4 days

Gaussian elimination/elementary matrices. Echelon and reduced echelon form. Existence/uniqueness of solutions. Matrix inverses. Row reduction interpreted as an LU-factorization.

III. Determinants — 2–3 days

Determinants are readily encountered when solving 2 by 2 and 3 by 3 general linear systems. The elementary properties of determinants are easily discovered or illustrated using the resulting expressions. Formal verifications in most cases should be avoided. Explore the uses of determinants as well as the difficulties in computing them. Main topics: cofactor expansion, determinants and row operations, $\det AB = \det A \det B$, and Cramer's Rule (to show the sensitivity of solutions to $Ax = b$).

IV. Properties of R^n **7–8 days**

Introduce R^n as a set of n-tuples and not as a formal vector space. Define vector addition and scalar multiplication, but it is not necessary to prove formally all the properties of vector addition and scalar multiplication. There should be a strong geometric emphasis in the presentation of this material.

1. Linear combinations: linear dependence and independence.
2. Bases of R^n.
3. Subspaces of R^n: spanning set, basis, dimension, row space and column space (range of A as a mapping), null space.
4. Matrices as linear transformations.
5. Rank: row rank = column rank, products, connections with invertible submatrices.
6. Systems of equations revisited: solution theory, rank + nullity = number of columns.
7. Inner product: length and orthogonality, orthogonal/orthonormal sets and bases, orthogonal matrices.

V. Eigenvalues and Eigenvectors **6 days**

Eigenvalues are important in a wide variety of applications. Sufficient time should be allowed for complete coverage of this topic. Eigenvectors may be introduced and/or motivated using geometric examples.

1. The equation $Ax = \lambda x$.
2. The characteristic polynomial and identification of some of its coefficients (trace, determinant), algebraic multiplicity of eigenvalues.
3. Eigenspaces, geometric multiplicity.
4. Similarity: distinct eigenvalues and diagonalization (with emphasis on $AP = PD$).
5. Symmetric matrices: orthogonal diagonalization, quadratic forms.

VI. More on Orthogonality **4 days**

Include the standard topics with a strong geometric emphasis: orthogonal projection onto a subspace; Gram-Schmidt orthogonalization and interpretation as a QR factorization; and the least square solutions of inconsistent linear systems, with applications to data-fitting.

TOTAL: 26–28 days

VII. Supplementary Topics

The following topics are frequently included in a beginning course. Choices will be influenced by time available, needs and interests of students, and course objectives:

Computational experience (see Recommendation #5).

Additional topics: abstract vector spaces, linear transformations, positive definite matrices, reduction of a symmetric matrix to a diagonal matrix by congruence, singular value decomposition, matrix norms.

Some applications, such as Markov chains, input-output models, Leslie matrices, difference equations, differential equations, linear programming.

3. Faculty should consider the needs and interests of students as learners.

Research in teaching and learning suggests several principles, including (1) a first linear algebra course should proceed from concrete and practical examples to general concepts (as proposed in Recommendation #2), and (2) students learn best, as we do, by active involvement—solving problems, making conjectures, and communicating with others. We encourage researchers, including mathematicians, to study issues such as teaching strategies, effective testing, abstraction, and the role of applications, as they apply to the teaching of linear algebra.

4. Faculty should be encouraged to utilize technology in the first linear algebra course.

We believe that student use of computers or supercalculators for homework and projects can reinforce concepts from lectures, contribute to the discovery of new concepts and make feasible the solution of realistic applied problems. Currently available personal computers are more than adequate for the computations required at this level (especially if they are equipped with a math coprocessor). Software available ranges from special purpose programs distributed with textbooks to commercial systems for both numeric and symbolic computation. Such software does not require computer programming experience.

5. At least one "second course" in matrix theory / linear algebra should be a high priority for every mathematics curriculum.

There are three main types of second courses: abstract vector spaces, matrix analysis/applied linear algebra, and numerical linear algebra (which may only partly fit this category). A second course in matrix analysis can provide as good a forum for rigor and mathematical aesthetics as a course on abstract vector spaces. It also provides notable utility, a basis for a broad range of graduate study, and opportunity for curricular efficiency. Departments with highly constrained upper level offerings may wish to offer different types of courses in alternate years. Finally, we recommend that the mathematics community may productively consider expanding the now traditional linear algebra course into a full year sequence, as is now common in other countries.

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Another Point of View...

THEOREM 1.5. *The ring* Hom(V, V) *is isomorphic to the ring of all n-by-n matrices with elements in k. The isomorphism depends on the choice of a basis. Let g be an element of* Hom(V, V) *which carries a selected new basis into the old one and suppose that* $g \to D$ *describes g in terms of the old basis. If* $f \to A$ *is the description of any f in terms of the old basis, the* DAD^{-1} *is the description of this same f in terms of the new basis.*

Mathematical education is still suffering from the enthusiams which the discovery of this isomorphism has aroused. The result has been that geometry was eliminated and replaced by computations. Instead of the intuitive maps of a space preserving addition and multiplication by scalars (these maps have an immediate geometric meaning), matrices have been introduced. From the innumerable absurdities—from a pedagogical point of view—let me point out one example and contrast it with the direct description.

Matrix method: A product of a matrix A and a vector X (which is then an n-tuple of numbers) is defined; it is also a vector. Now the poor student has to swallow the following definition:

A vector X is called an eigen vector if a number λ exists such that

$$AX = \lambda X.$$

Going through the formalism, the characteristic equation, one then ends up with theorems like: If a matrix A has n distinct eigen values, then a matrix D can be found such that DAD^{-1} is a diagonal matrix.

The student will of course learn all this since he will fail the course if he does not.

Instead one should argue like this: Given a linear transformation f of the space V into itself. Does there exist a line which is kept fixed by f? In order to include the eigen value 0 one should then modify the question by asking whether a line is mapped *into* itself. This means of course for a vector spanning the line that

$$f(X) = \lambda X.$$

Having thus motivated the problem, the matrix A describing f will enter only for a moment for the actual computation of λ. It should disappear again.

E. Artin, *Geometric Algebra*, Interscience, New York, 1957, p. 13.