Abstract

An $n \times n$ matrix A_c is called W_c-companion matrix if A_c is defined by

$$A_c = \begin{bmatrix} 0 & W_c \\ a_0 & \mathrm{d}^* \end{bmatrix}$$

with $a_0 \neq 0,\ \mathrm{d}^* = (a_1, \ldots, a_{n-1})$ and $W_c W_c^* = W_c^* W_c = cI_{n-1}$ for some positive real number c. A companion matrix is a special case of W_c-compaion matrix. We give the explicit spectral norm of W_c-companion matrix and bounds for eigenvalues of W_c-companion matrix. And, an explict formula for the polar decomposition of an $n \times n$ W_c-companion matrix is derived.

Explicit Spectral Norm and Polar Decomposition of W_c-companion Matrices

Gwang-Yeon Lee
Department of Mathematics
Hanseo University
Seosan, 356-820, Korea

Jin-Soo Kim and Sang-Gu Lee
Department of Mathematics
Sungkyunkwan University
Suwon, 440-746 Korea

Seok-Zun Song
Department of Mathematics
Cheju National University
Cheju 690-756, Korea

February 12, 2001

1 Introduction

We consider a monic complex polynomial

$$p(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0, \quad a_0 \neq 0.$$

Is there a matrix A whose $p(z)$ is the minimal polynomial is $p(z)$? If so, the size of the matrix A must be at ast n; it is not hard to find such a matrix $A \in M_n(\mathcal{C})$ when $\mathcal{C}$ is a complex field. Consider the matrix

$$C = \begin{bmatrix} 0 & I_{n-1} \\ a_0 & \mathrm{d}^* \end{bmatrix},$$

*This paper was supported by the Basic Science Research Institute Program, Ministry of Education, Project No. BSRI-98-1406, 1420 and by TGRC-KOSEF

with $a_0 \neq 0$ and $\mathrm{d}^* = (a_1, a_2, \ldots, a_{n-1})$. We call the matrix C is an n by n nonsingular companion matrix associated with $p(z)$.

Now, we define a W_c-companion matrix. For a positive real number c, an $(n-1) \times (n-1)$ complex matrix W_c satisfying $W_c W_c^* = W_c^* W_c = cI_{n-1}$ where I_{n-1} is the identity matrix of order $n-1$. We call the $n \times n$ matrix A_c W_c-*companion matrix* if A_c is defined by

$$A_c = \begin{bmatrix} 0 & W_c \\ a_0 & \mathrm{d}^* \end{bmatrix} \tag{1}$$

with $a_0 \neq 0$, $\mathrm{d}^* = (a_1, \cdots, a_{n-1})$. If W_c is the identity matrix, then A_c is a companion matrix. So, the matrix A_c is a general form of the companion matrix C.

Let B be an $m \times n$ matrix. The *singular values* $\sigma_i(B)$ of B are nonnegative square roots of the eigenvalues of the positive semidefinite matrix BB^*, or equivalently, they are the eigenvalues of the positive semidefinite square root $(BB^*)^{\frac{1}{2}}$, so that

$$\sigma_i(B) = [\lambda_i(BB^*)]^{\frac{1}{2}} = \lambda_i[(BB^*)^{\frac{1}{2}}], \quad i = 1, 2, \cdots, m.$$

The singular values are real and nonnegative. The vector of singular values arranged in nonincreasing order is denoted by $\sigma(B) = (\sigma_1(B), \cdots, \sigma_m(B))$. For a positive semidefinite Hermitian matrix B, $\sigma_i(B) = \lambda_i(B)$, $i = 1, 2, \cdots, n$. Unfortunately, in general, a W_c-companion matrix A_c is not positive semidefinite Hermitian matrix.

Let $C = PU$ be the left polar decomposition of C with positive definite P and unitary U. In [VW], P. van den Driessche and H. K. Wimmer yeild the explicit formula for P and U in the left polar decomposition of a nonsingular companion matrix C where the coefficients of the polynomial $p(z)$ form the last row. Our natural question now becomes, what is the explicit polar decompositon formula of

$$\tilde{C} = \begin{bmatrix} 0 & W \\ a_0 & \mathrm{d}^* \end{bmatrix}$$

for a permutation matrix W, for an orthogonal matrix W, for a unitary matrix W or, more general, for $W_c W_c^* = W_c^* W_c = cI_{n-1}$?

In section 2, we give explicit spectral norm and consider bounds for eigenvalues of a W_c-companion matrix. In section 3, we yeild the explicit formula for P and U in the left polar decomposition of a W_c-companion. In section 4, we give explicit spectral norm of a matrix G which is defined

$$G = \begin{bmatrix} X & I_{n-1} \\ a_0 & \mathrm{d}^* \end{bmatrix},$$

where $X = (c, 0, \ldots, 0)^T$.

2 Spectral Norm of W_c-companion Matrices

Let B be an $n \times n$ matrix. The spectral norm of B defined by

$$|||B|||_2 \equiv \max\{\sqrt{\lambda} \ : \ \lambda \text{ is an eigenvalue of } BB^*\}.$$

Notice that if $B^*B\mathrm{x} = \lambda\mathrm{x}$ and $\mathrm{x} \neq 0$, then $\mathrm{x}^*B^*B\mathrm{x} = ||B\mathrm{x}||_2^2 = \lambda||\mathrm{x}||_2^2$, so $\lambda \geq 0$. Therefore $\sqrt{\lambda}$ is nonnegative real.

In this section, we give explicit spectral norm for the the matrix of type (1) and give bounds for eigenvalues of W_c-companion matrix.

Let A_c be an $n\times n$ W_c-companion matrix for a positive real number c. Then

$$A_cA_c^* = \begin{bmatrix} cI_{n-1} & W_c\mathrm{d} \\ (W_c\mathrm{d})^* & s \end{bmatrix} \tag{2}$$

where $s = \sum_{i=0}^{n-1} |a_i|^2 = |a_0|^2 + ||\mathrm{d}||^2$.

Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of A_c and let

$$\mathcal{D} = \{(x_1, x_2, \ldots, x_n) \in R^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

Lemma 2.1 *Let $(\sigma_1^2, \ldots, \sigma_n^2) \in \mathcal{D}$. Then $\sigma_2^2 = \cdots = \sigma_{n-1}^2 = c$, and σ_1^2, σ_n^2 are roots of $f(z) = z^2 - (s+c)z + c|a_0|^2$.*

Proof Since

$$A_cA_c^* = \begin{bmatrix} cI_{n-1} & W_c\mathrm{d} \\ (W_c\mathrm{d})^* & s \end{bmatrix},$$

we can easily verify that

$$\begin{aligned} \det(zI_n - A_cA_c^*) &= (z-s)\det[(z-c)I_{n-1}] - (W_c\mathrm{d})^*\mathrm{adj}[(z-c)I_{n-1}](W_c\mathrm{d}) \\ &= (z-c)^{n-2}[z^2 - (s+c)z + cs - c||\mathrm{d}||^2] \\ &= (z-c)^{n-2}(z^2 - (s+c)z + c|a_0|^2) \end{aligned}$$

Thus $A_cA_c^*$ has c as the eigenvalue of multiplicity at least $(n-2)$. Since the eigenvalues of the principal submatirx cI_{n-1} in (2) interlace with those of $A_cA_c^*$, it follows that $\sigma_n^2 \leq c \leq \sigma_1^2$ and σ_1^2, σ_n^2 are roots of $f(z)$. ■

Since σ_1^2 and σ_n^2 are roots of $f(z)$, $\sigma_1\sigma_n = \sqrt{c}|a_0|$ and $\sigma_1^2 + \sigma_n^2 = s + c$. Since

$$||\mathrm{d}||^2 = s - \frac{\sigma_1^2\sigma_n^2}{c} = -\frac{1}{c}(\sigma_1^2 - c)(\sigma_n^2 - c),$$

we have $\sigma_1 + \sigma_n = \alpha$ and $||\mathrm{d}||^2 = (\alpha + |a_0| + \sqrt{c})(\alpha - |a_0| - \sqrt{c})$, where

$$\alpha = \left(s + 2\sqrt{c}|a_0| + c\right)^{\frac{1}{2}}. \tag{3}$$

Set $\beta = (s - 2\sqrt{c}|a_0| + c)^{\frac{1}{2}}$. Then the following theorem holds.

Theorem 2.2 *Let A_c be an $n \times n$ W_c-companion matrix as in (1). Then*

$$|||A_c|||_2 = \sigma_1 = \frac{\alpha+\beta}{2}.$$

Proof By Lemma 2.1, the eigenvalues of $A_c A_c^*$ are c and $\frac{(s+c)\pm\sqrt{(s+c)^2-4c|a_0|^2}}{2}$. Since $||\mathrm{d}||^2 \geq 0$, $\sigma_1 \geq \sqrt{c}$ and hence

$$|||A_c|||_2 = \sigma_1 \quad = \quad \sqrt{\frac{(s+c)+\sqrt{(s+c)^2-4c|a_0|^2}}{2}} = \frac{\alpha+\beta}{2}.$$

■

Note that the spectral norm of A_1 does not depend on W_1. In particular, if $a_0 = 1$ and $\mathrm{d}^* = (1, 1, \ldots, 1)$ in A_1 then

$$|||A_1|||_2 = \frac{\sqrt{n+3}+\sqrt{n-1}}{2}.$$

Let $p(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0, \quad a_0 \neq 0$. If $\tilde{z}$ is a root of $p(z) = 0$ and if $|||\cdot|||$ is any matrix norm on M_n, then $|\tilde{z}| \leq |||C|||$ where C is an $n \times n$ companion matrix associated matrix $p(z)$. Let

$$q(z) = -\frac{1}{a_0} z^n p(\frac{1}{z}).$$

Then

$$q(z) = z^n - \frac{a_1}{a_0} z^{n-1} - \frac{a_2}{a_0} z^{n-2} - \cdots - \frac{a_{n-1}}{a_0} z - \frac{1}{a_0}$$

and the roots of $q(z) = 0$ are exactly reciprocals of the roots of $p(z) = 0$. So, we have the following corollary. This corollary is about a lower bound of the roots of $p(z) = 0$.

Corollary 2.3 *Let $\tilde{z}$ be a root of the characteristic equation of a companion matrix C. Then*

$$|\tilde{z}| \geq \frac{\alpha-\beta}{2},$$

where $\alpha = (s+1+2|a_0|)^{\frac{1}{2}}$ and $\beta = (s+1-2|a_0|)^{\frac{1}{2}}$.

Proof Since $q(z) = -\frac{1}{a_0} z^n p(\frac{1}{z})$,

$$\left|\frac{1}{\tilde{z}}\right| \leq \frac{\alpha+\beta}{2|a_0|}.$$

Thereofore, $|\tilde{z}| \geq \frac{\alpha-\beta}{2}$. ■

We now consider eigenvalues of W_c-companion matrix A_c. For x, y $\in \mathcal{D}$, x $\prec$ y if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$, $k = 1, 2, \ldots, n$ and equality holds when $k = n$. When x $\prec$ y, x is said to be *majorized* by y. For x, y $\in \mathcal{D}$, x $\prec_w$ y if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$, $k = 1, 2, \ldots, n$. When x $\prec_w$ y, x is said to be *submajorized* by y. The following is an interesting simple fact; $(\bar{x}, \ldots, \bar{x}) \prec (x_1, \ldots, x_n)$, where $\bar{x} = (\sum_{i=1}^n x_i)/n$.

A real valued function ϕ defined on a set $F \subseteq R^n$ is said to be *Schur-convex* on F if x $\prec$ y $\implies \phi(\mathrm{x}) \leq \phi(\mathrm{y})$, and *Schur-concave* on F if x $\prec$ y $\implies \phi(\mathrm{x}) \geq \phi(\mathrm{y})$.

Denote by $S_k(\mathrm{x})$ the kth *elementary sysmmetric function* of x $= (x_1, \ldots, x_n)$. That is,

$$S_0(\mathrm{x}) \equiv 1, \quad S_1(\mathrm{x}) = \sum_{i=1}^n x_i, \quad S_2(\mathrm{x}) = \sum_{i<j} x_i x_j,$$

$$S_3(\mathrm{x}) = \sum_{i<j<k} x_i x_j x_k, \ \ldots, \ S_n(\mathrm{x}) = \prod_{i=1}^n x_i.$$

Then, it is a well known fact that the elementary symmetric funtion is Schur-concave .

Using this elementary symmetric function, we have the following theorem.

Theorem 2.4 *Let A_c be an $n \times n$ W_c-companion matrix and let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A_c ordered so that $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) \in \mathcal{D}$. Then*

$$(|a_0|c^{\frac{n-1}{2}})^{\frac{1}{n}} \leq |\lambda_1|.$$

Proof Since $|a_0|c^{\frac{n-1}{2}} \neq 0$, $\det(A_c) = (-1)^{n+1} a_0 c^{\frac{n-1}{2}} = \lambda_1 \lambda_2 \cdots \lambda_n \neq 0$. So, $|\lambda_1| \cdots |\lambda_n| = |a_0|c^{\frac{n-1}{2}} > 0$ and $(\bar{\lambda}, \ldots, \bar{\lambda}) \prec (|\lambda_1|, \ldots, |\lambda_n|)$, where $\bar{\lambda} = (\sum_{i=1}^n |\lambda_i|)/n$. Since the nth elementary symmetric function S_n is Schur-concave,

$$|a_0|c^{\frac{n-1}{2}} = |\lambda_1| \cdots |\lambda_n| \leq \left(\frac{\sum_{i=1}^n |\lambda_i|}{n} \right)^n.$$

So,

$$n(|a_0|c^{\frac{n-1}{2}})^{\frac{1}{n}} \leq \sum_{i=1}^n |\lambda_i|.$$

Thus,

$$\left((|a_0|c^{\frac{n-1}{2}})^{\frac{1}{n}}, \ldots, (|a_0|c^{\frac{n-1}{2}})^{\frac{1}{n}} \right) \prec_w (|\lambda_1|, \ldots, |\lambda_n|).$$

Therefore,

$$(|a_0|c^{\frac{n-1}{2}})^{\frac{1}{n}} \leq |\lambda_1|.$$

■

Note that for any $n \times n$ nonsingular complex matrix B,

$$(|\lambda_1(B)|, \ldots, |\lambda_n(B)|) \prec_w (\sigma_1(B), \ldots, \sigma_n(B)).$$

Then, for an $n\times n$ matrix A_c, $(|\lambda_1|, \ldots, |\lambda_n|) \prec_w (\sigma_1, \ldots, \sigma_n)$ and hence $\sum_{i=1}^n |\lambda_i| \leq (n-2)\sqrt{c} + \sigma_1 + \sigma_n$.

Therefore,

$$n(|a_0|c^{\frac{n-1}{2}})^{\frac{1}{n}} \leq \sum_{i=1}^n |\lambda_i| \leq (n-2)\sqrt{c} + \sqrt{s + c + 2\sqrt{c}|a_0|}.$$

We now give bounds on eigenvalues and singular values of A_c.

Lemma 2.5 *[MO, E.1.b] For any nonsingular complex matrix A of order n,*

$$(|\lambda_1(A)|^2, \cdots, |\lambda_n(A)|^2) \prec (\sigma_1^2(A), \cdots, \sigma_n^2(A)),$$

where $\lambda_i(A)$ are eigenvalues of A and $\sigma_i(A)$ are singular values of A, $i = 1, 2, \cdots, n$.

Theorem 2.6 *Let A_c be an $n \times n$ W_c-companion matrix and let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A_c ordered so that $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) \in \mathcal{D}$. Then*

$$\sigma_n \leq |\lambda_n| \leq \sqrt{\frac{(n-1)c+s}{n}} \leq |\lambda_1| \leq \sigma_1$$

Proof Since $||\mathrm{d}||^2 > 0$, by Lemma 2.5,

$$\sigma_n^2 \leq |\lambda_n|^2, \quad |\lambda_1|^2 \leq \sigma_1^2 \tag{4}$$

Since $\sum_{i=1}^n \sigma_i^2 = (n-1)c + s$ and $c > 0$,

$$\left(\frac{(n-1)c+s}{n}, \ldots, \frac{(n-1)c+s}{n}\right) \prec (|\lambda_1|^2, \ldots, |\lambda_n|^2).$$

So,

$$|\lambda_n|^2 \leq \frac{(n-1)c+s}{n} \leq |\lambda_1|^2 \tag{5}$$

Therefore, by (4) and (5),

$$\sigma_n \leq |\lambda_n| \leq \sqrt{\frac{(n-1)c+s}{n}} \leq |\lambda_1| \leq \sigma_1.$$

■

3 Polar decomposition of a W_c-companion matrices

Let A be an $m \times n$ matrix, $m \leq n$. Then A may be written as $A = PU$ where P is positive semidefinite whose rank is the same as that of A, and U is unitary. In this section, our main goal is to give the explicit left polar decomposition of a W_c-companion matrix A_c.

Now, we yield the explicit left polar decomposition of a W_c-companion matrix A in (1). The following theorem is the explicit formula for a positive definite P, and a unitary matrix U in the left polar decomposition of a W_c-companion matrix.

Set $\delta = \alpha\sqrt{c} + |a_0|\sqrt{c} + c$ and set

$$\Gamma = \sqrt{c}\alpha I_{n-1} - \delta^{-1}(W_c\mathrm{d})(W_c\mathrm{d})^*.$$

Theorem 3.1 *Let A_c be an $n \times n$ W_c-companion matrix. Then*

$$P = \frac{1}{\alpha}\begin{bmatrix} \Gamma & W_c d \\ (W_c d)^* & s + \sqrt{c}|a_0| \end{bmatrix} \tag{6}$$

is positive definite, $P = (A_c A_c^*)^{\frac{1}{2}}$ and $A_c = PU$ is the left polar decomposition of A_c for a unitary matrix

$$U = \frac{1}{\alpha}\begin{bmatrix} \frac{a_0}{|a_0|^2}(-\frac{\alpha}{\sqrt{c}} + \delta^{-1}||\mathrm{d}||^2 + 1)W_c\mathrm{d} & W_c(\frac{\alpha}{\sqrt{c}}I_{n-1} - \delta^{-1}\mathrm{d}\mathrm{d}^*) \\ \frac{a_0}{|a_0|}(\sqrt{c} + |a_0|) & \mathrm{d}^* \end{bmatrix}. \tag{7}$$

Proof For a positive definite matrix P and a unitary matrix U, $A_c = PU$ is left polar decomposition of A_c. So, $P = (A_c A_c^*)^{\frac{1}{2}}$.

First, we consider the case with $\sigma_1^2 = c$ or $\sigma_n^2 = c$. This is equivalent to $\mathrm{d} = 0$, by Lemma 2.1. In this case $\Gamma = \sqrt{c}\alpha I_{n-1}$ and

$$\begin{aligned} P &= (A_c A_c^*)^{\frac{1}{2}} = \mathrm{diag}(\sqrt{c}, \cdots, \sqrt{c}, |a_0|) \\ &= \frac{1}{\alpha}\begin{bmatrix} \Gamma & 0 \\ 0 & |a_0|^2 + \sqrt{c}|a_0| \end{bmatrix}. \end{aligned}$$

Since $A_c = PU$ with P as above,

$$U = \begin{bmatrix} 0 & \frac{1}{\sqrt{c}}W_c \\ \frac{a_0}{|a_0|} & 0 \end{bmatrix}.$$

In the case $\sigma_1^2 < c < \sigma_n^2$, that is $\mathrm{d} \neq 0$. Let

$$\mathrm{v}_1 = \frac{1}{\sqrt{c}\,||\mathrm{d}||}\begin{bmatrix} W_c\mathrm{d} \\ 0 \end{bmatrix},$$

and let $\mathrm{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ be an $n \times 1$ matrix with $n-1$ zeros. Then, $A_cA_c^*\mathrm{v}_1 = c\mathrm{v}_1 + \sqrt{c}\,||\mathrm{d}||\,\mathrm{v}_2$ and $A_cA_c^*\mathrm{v}_2 = \sqrt{c}\,||\mathrm{d}||\,\mathrm{v}_1 + s\mathrm{v}_2$. So,

$$A_cA_c^*(\mathrm{v}_1, \mathrm{v}_2) = (\mathrm{v}_1, \mathrm{v}_2)\begin{bmatrix} c & \sqrt{c}\,||d|| \\ \sqrt{c}\,||d|| & s \end{bmatrix}.$$

For the computation of the square root of $A_cA_c^*$, it is sufficient to consider a symmetric 2×2 matrix. Let

$$H = \begin{bmatrix} c & \sqrt{c}||\mathrm{d}|| \\ \sqrt{c}||\mathrm{d}|| & s \end{bmatrix}.$$

Then $\det(zI - H) = f(z)$ and

$$H^{\frac{1}{2}} = \frac{\sqrt{c}}{\alpha}\begin{bmatrix} \sqrt{c} + |a_0| & ||\mathrm{d}|| \\ ||\mathrm{d}|| & \frac{\alpha^2}{\sqrt{c}} - |a_0| - \sqrt{c} \end{bmatrix}.$$

Now, we consider the eigenvalue c of $A_cA_c^*$. Let $\mathrm{y}_2, \cdots, \mathrm{y}_{n-1}$ be an orthonormal set of eigenvectors of $A_cA_c^*$ satisfying $A_cA_c^*\mathrm{y}_i = c\mathrm{y}_i$ for $i = 2, \ldots, n-1$. Then for each y_i we have $\mathrm{y}_i^* = (\mathrm{x}_i^*, 0)$ and $(W_c\mathrm{d})^*\mathrm{x}_i = 0$. So, the matrix $V = (\mathrm{y}_2, \cdots, \mathrm{y}_{n-1}, \mathrm{v}_1, \mathrm{v}_2)$ is a unitary matrix, and

$$V^*A_cA_c^*V = \begin{bmatrix} cI_{n-2} & 0 \\ 0 & H \end{bmatrix}$$

Since P is positive definite, each eigenvalue of P is positive and hence

$$P = (A_cA_c^*)^{\frac{1}{2}} = \left(V\begin{bmatrix} cI_{n-2} & 0 \\ 0 & H \end{bmatrix}V^*\right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} P &= \sqrt{c}I_n + (\mathrm{v}_1, \mathrm{v}_2)(H^{\frac{1}{2}} - \sqrt{c}\,I_2)\begin{bmatrix} \mathrm{v}_1^* \\ \mathrm{v}_2^* \end{bmatrix} \\ &= \frac{1}{\alpha}\begin{bmatrix} \Gamma & W_c\mathrm{d} \\ (W_c\mathrm{d})^* & s + \sqrt{c}\,|a_0| \end{bmatrix}. \end{aligned}$$

Since A_c is nonsingular,

$$A_c^{-1} = \begin{bmatrix} -\frac{(W_c\mathrm{d})^*}{c\,a_0} & \frac{1}{a_0} \\ \frac{1}{c}W_c^* & 0 \end{bmatrix}.$$

For the unitary factor of $A_c = PU$ we have $U = P(A_c^{-1})^*$. Hence

$$U = \frac{1}{\alpha}\begin{bmatrix} \frac{a_0}{|a_0|^2}(-\frac{\alpha}{\sqrt{c}} + \delta^{-1}||\mathrm{d}||^2 + 1)W_c\mathrm{d} & W_c(\frac{\alpha}{\sqrt{c}}I_{n-1} - \delta^{-1}\mathrm{d}\mathrm{d}^*) \\ \frac{a_0}{|a_0|}(\sqrt{c} + |a_0|) & \mathrm{d}^* \end{bmatrix}.$$

The proof is completed. ■

4 Other Result

Norms may be thought of as generalizations of Euclidean length, but the study of norms is more than exercise in mathematical generalization. It is necessary for a proper formulation of notions such as power series of matrices, and it is essential in the analysis and assessment of algorithms for numerical computations. But, to compute of any matrix norm is very hard problem.

Now, we consider an $n \times n$ matrix G defined by

$$G = \begin{bmatrix} X & I_{n-1} \\ a_0 & \mathrm{d}^* \end{bmatrix}, \tag{8}$$

where $a_0 \neq 0$, $\mathrm{d}^* = (a_1, \ldots, a_{n-1})$ and $X = (c, 0, \ldots, 0)^T$.

Set $e = s + |c|^2 + 2$, $k = (|c|^2 + 1)(s + 1) - |\bar{a}_0 c + a_1|^2 + |a_1|^2 + |a_0|^2$, and $l = 2(|c|^2 + 1)s - |ca_1 - \bar{a}_0|^2$. Then the following theorem holds and hence we can get the sprectral norm for G by calculated cubic equation.

Theorem 4.1 *Let G be an $n \times n$ matrix up to permutation as in (8). Then*

$$|||G|||_2 = \max\{\sqrt{\lambda} \mid \lambda = 1,\ \lambda^3 - e\lambda^2 + k\lambda + l = 0\}.$$

Proof Since

$$GG^* = \begin{bmatrix} \mathrm{diag}(|c|^2 + 1, 1, \ldots, 1) & \bar{a}_0 X + \mathrm{d} \\ (\bar{a}_0 X + \mathrm{d})^* & s \end{bmatrix},$$

$$\begin{aligned} \det(\lambda I_n - GG^*) &= (\lambda - s)(\lambda - (|c|^2 + 1))(\lambda - 1)^{n-2} \\ &\quad -(\bar{a}_0 X + \mathrm{d})^* \mathrm{adj}(\mathrm{diag}(|c|^2 + 1, 1, \ldots, 1))(\bar{a}_0 X + \mathrm{d}) \\ &= (\lambda - 1)^{n-3}[(\lambda - s)(\lambda - (|c|^2 + 1))(\lambda - 1)^{n-2} - |\bar{a}_0 c + a_1|^2(\lambda - 1) \\ &\quad -(|a_2|^2 + \cdots + |a_{n-1}|^2)(\lambda - (|c|^2 + 1))] \\ &= (\lambda - 1)^{n-3}(\lambda^3 - e\lambda^2 + k\lambda + l). \end{aligned}$$

The proof is completed. ■

References

[HJ] R. A. Horn and C. R. Johnson, *Matrix Analysis,* Cambridge Univ. Press, 1985.

[LLS] G.-Y. Lee, S.-G. Lee and H.-G. Shin, On the k-generalized Fibonacci Matrix Q_k, *Linear Algebra and Its Appl.*, 251:73-88, 1997.

[MO] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic, New York, 1979.

[VW] P. van den Driessche and H. K. Wimmer, Explicit Polar Decomposition of Companion Matrices. *The Electronic J. Linear Alg.*, 1:64-69,1996.