TSG 14: Innovative approaches to the teaching of mathematics

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Aims and Focus

The aim of this group is to present some methods for teaching mathematics which can be described as ‘innovative’ when compared to exposition-and-exercise, dominant use of textbook or programmed published materials. In the sessions, participants will experience methods for themselves, perhaps by doing some mathematics, and discuss the nature of learning in such environments. The methods presented will be generic, that is they can be used with a wide range of topics, and the presentations will relate them closely to specific content. The group aims to be a showcase for ‘cutting-edge’ thinking about ways of teaching mathematics which may take a range of learning theories into account. The emphasis will be on the learning possibilities presented by pedagogic techniques. There will be no paper presentations, nor will it be a showcase for commercial materials, 'tips for teachers', or individual lesson ideas. Papers which contribute to this focus will be reviewed and, if accepted, displayed on the website and/or distributed at the sessions. The possibility of a publication will be discussed at the conference.

Call for Papers

Papers (which should be written in English) will be reviewed and accepted for 'presentation by distribution' on the website and/or at the sessions, subject to the authors' attendance. The final date for submission of papers will be March 1st (note this is later than the date proposed by the ICME organisers). They should be sent to Anne Watson in electronic form and will be reviewed by the TSG organising team. Papers should address the aims and focus described above, and in addition should refer to underlying theory on which the teaching is based, and evidence from practice and/or research about its value. Papers should be a maximum of 5 pages in 12pt Times. Further guidelines about style will be issued to those whose papers are accepted. Note that there has already been considerable interest in this group, so even if a paper conforms to the aims and is well-written and scientifically sound, we may still have to restrict what we accept in order to represent the whole field. **We are particularly interested in receiving papers about work and projects which have not been extensively reported in research journals.** Potential authors are invited to contact us early with an outline of their proposed paper if they would like early feedback.

Practical Information

Please contact the group organiser for information

Programme

The programme of presentations to take place during the sessions is already at an advanced stage of planning. We are interested in hearing about outstanding work and presenters who might fit with our aims, but already have more than enough to fill the time. More details will be given in due course but it is expected that there will be presentations of teaching methods relating to mathematical thinking, investigation, chanting, discussion, deep understanding, interactive techniques which relate to mathematical structure, and so on …. from a variety of countries. There will be time for discussion of issues arising.

Papers and Discussion Documents

Author	Title - download as pdf
Anne Watson	General introduction ot the papers
Karin Brodie	Teaching mathematics and social justice: multidimensionality and responsibility
Denise Grenier	Research situations for teaching: a modelling proposal and example
Gary Flewelling	COMPARING THE GAMES WE PLAY IN THE CLASSROOM
Cécile Ouvrier-Buffet	“RESEARCH” AS A WAY OF LEARNING NEW CONCEPTS: THE PARTICULAR CASE OF CONSTRUCTING DEFINITIONS
Binyan Xu	Mathematic Instructional Design in Innovative Learning Perspectives, A Case Study on Symmetry Learning: Elementary Framework and Initial Progress
Megan Staples	Using Common Ground to Understand Innovative Teaching
John Mason	A PHENOMENAL APPROACH TO MATHEMATICS
Marcus Vinicius Maltempi	LEARNING VORTEX, GAMES AND TECHNOLOGIES: A NEW APPROACH TO THE TEACHING OF MATHEMATICS
Xu Liquan	Briefing on MM Education Way, a New Way of Mathematics Teaching
Khoon Yoong WONG	Using Multi-Modal Think-Board to Teach Mathematics
Anne Watson	Dance and mathematics: power of novelty in the teaching of mathematics
Asuman Duatepe	DRAMA BASED INSTRUCTION AND GEOMETRY
Malcolm Swan	Developing mathematics lesson genres
Luo Qiu jia	The Open-Ended Approach in Reforming Traditional Teaching: taking learning plane geometry as an example

Walter G Spunde	Functions as First Class Citizens

Please contact the group organiser for further information

Research situations for teaching: a modelling proposal and example

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Laboratoire Leibniz, University Joseph Fourier, Grenoble

Scientific knowledge is built in the context of research, especially through solving open questions. This observation led us to study how didactic processes could exist and work around research situations. The training involved is concerned primarily with "transverse" knowledge, i.e. those which play a role in many scientific fields, such as experimentation, conjectural statements, argumentation, modeling, definitions, proofs, implications, structuring, decomposition-recomposition, induction. However, we observed that, one the one hand, learning this transverse knowledge is a constant objective which has been declared "essential" after several program reforms in secondary education in France, and that, on the other hand, there is an intrinsic difficulty to carry out these objectives in class.

The types of situations which we analyze here have been worked out for a long time in various workshops, at all school levels, and have been studied from a theoretical point of view in the last three years by the SIRC group, composed of researchers from various departments and teachers from secondary education.

I. Research situations for teaching : a modelling

I.1. Hypotheses and research questions

In order to make progress in a research situation, a researcher can, and must often, select by himself a suitable framework of resolution, must modify the rules or allow himself to redefine objects or questions. This is precisely this type of practices which we wish to get pupils involved with, because they are the foundation of mathematical activity. However it seems that this type of practices is not usual in class, and even that it is practically forbidden in many circumstances.

This raises the question of finding which conditions are needed in didactic institutions to create a mathematical activity which pertains to a "research situation", and is likely to allow the learning of what we have called "transverse" knowledge.

I.2. A model of "Research Situation for the Class" (RSC)

For us, an RSC must fulfill the following criteria (These criteria will be developed in the session).

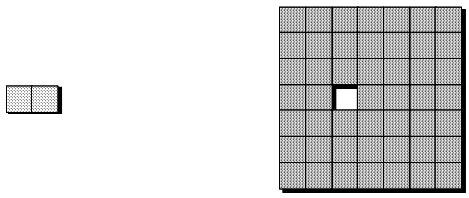
1. *A RSC is akin to a professional research strategy*. It must be related in some way to unsolved questions. Because, there is a strong argument that a close contact to unsolved questions, not only for the pupils, but also for teachers and researchers, will be decisive for establishing the pupils' positioning with respect to the situation.
2. *The initial question should be of an easy access*. In particular, the question can be easily understood by pupils only when the problem does not require heavily formalized mathematics.
3. *Possible initial strategies are in view*, without requiring specific prerequisites.
4. *Several research strategies and several developments are possible*, from the point of view of mathematical activity (construction, proof, calculation) as well as from the point of view of mathematical concepts involved.
5. *A solved question can possibly lead to other new questions* .

I.3. Illustration of this model on a particular situation

This situation (Grenier & Payan (1998)) has been tested for several years, from the CM1 level (9 year old pupils) to University (DEA, i.e. first year of Post Graduate studies) and is now integrated regularly in various teaching curricula.

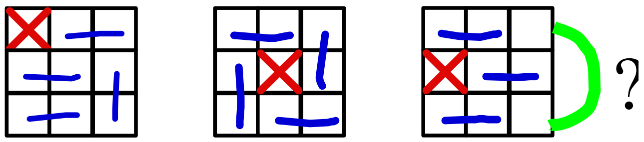
The proposed problem is a paving problem, namely paving a certain domain of boxes by pieces, without overlaps or overflows beyond the limits. More precisely, the question consists of knowing whether a given "polymino" can be paved by copies of identical smaller polyminos. In this generality, it is an open question which stands no chance of being solved. Researchers are currently interested in rather particular cases, for instance, whether it is possible to pave subsets of a square grid by smaller polyminos.This question is of an easy access, even for very young pupils.

Pupils and teachers first agree on a starting point of research, for example the following question : can one pave with dominos, a square grid from which a single cell has been removed (this cell being arbitrary). Here is an example as it could be shown to pupils.

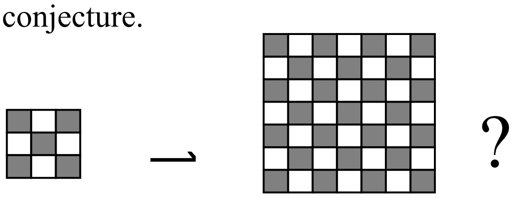


Initial strategies exist. It appears sufficient to have an intuitive feeling of space allowing to identify a set of cells and to understand what is a paving; this knowledge is already available at the nursery school level. The concept of parity is of course involved, but it is not essential, in fact the situation is a tool for investigating and understanding it in more depth.

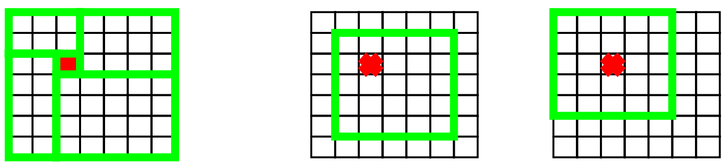
Devolution of the problem is immediate: some testing of pavements around the initial example allows to get a grasp on the question, without solving it completely. When the point is to advance further in the problem, it becomes necessary to change the stated question: one will for instance work with smaller squares (3x3 or 5x5). When pupils do not manage to pave, they can notice that in order to cover certain cells some choices are forced; one therefore obtains a proof of impossibility through "forced choices".



There are various strategies for progressing in the research. The location, on a 3x3 grid (and also on a 5x5 grid), of positions of cells which are to be removed to allow paving (represented in gray on the figure), induces a more general conjecture.



Evidence on the possibility of paving is given by various ways of partitioning, among which inductive procedures play a role ("inductive partitioning"). Examples :



Proofs of impossibility are obtained by structuring the object, while exploiting the form colored by the "right" boxes: since a domino covers a "black" cell and a "white" cell, a pavable polymino must necessarily be "balanced" (i.e. it must contain as many black cells as white boxes in a "check board" coloring).

The situation is not a "dead end": in fact, the phase of resolution of the initial problem (paving of a square grid with one cell removed) has raised in a natural way further questions.

I.4. Knowledge and environment concerned by a RSC

Our experimental research confirms that there is actual training involved, and that these are constitutive of any mathematical research activity: arguing, conjecturing, structuring objects, proving, modeling - all these items are more or less involved according to the selected RSC.This is what gives an institutional legitimacy to these situations.

The elements of the triple (question, conjecture, proof) are the "invariants" of the RSC. The associated "didactic variables" are "variables of research", in the sense that they determine the understanding and the interest of the question, its suitability for opening new questions, broadening research strategies, transforming the problem (modeling).

The criterion of success for pupils is not only, as in usual exercises, to solve the question (whether the solution is right or false). A "partial" criterion of success can be that pupils have raised a strong conjecture, or simply solved a particular case. The criterion of success for the teacher is the recognition of progress in the area of procedural knowledge (question, conjecture, proof).

I.5. Position of the actors in the didactic situation comprising a RSC

In a RSC, the actors (pupils and teachers) are in positions which differ from the usual ones of traditional didactic situation.

• Pupils are in a researcher position because they are assigned the task of producing something "new" which is not only new for them. Our experimental data show that, for pupils, the fact of knowing that they are trying to solve an unsolved, or only partially solved, problem, modifies their approach to the activity.

• Teachers are in the combined position of researchers and managers of the situation. As researchers, their position is closer to the pupils than in a traditional situation. But they are (supposedly) possessing the required transverse knowledge and the evaluation criteria for their validity. The "institutional relationship" between pupils and teachers is indeed concerned with this transverse knowledge. The corresponding basic rules are the usual ones occurring in "scientific debate" (Legrand, 1993).

II. An experimental situation

Part of our research is focused on studying research situations presented in the form of games[1], and introduced using suitable material support. We make the assumption that such a presentation can be a help with the devolution of the problem, already at the primary school level. Thus, through experiments which we carry out, we try to answer the following questions:

What is the role of the material gaming medium in the devolution of the research situations ?

What can be the influence of this medium on the research strategies ?

How can one manage a RSC presented in the form of a material game ?

II.1. An example: the wheel with colored pawns

Formulation of the task.

A fair organiser proposes a game made up of two discs of different sizes, laid out in a concentric way. On the largest disc, he displays a certain number of pawns, all of different colors.

Rules*:* The player must place on the small disc the same number of pawns as on the large disc. These pawns can use one, two, three, four or more colors, selected among the colors laid out on the large disc by the organiser. The small disc turns, notch by notch. The player wins if, in each position of the small disc, one and only one of its pawns is of the same color as that which corresponds to it on the large disc. How can the player choose and lay out his pawns to win? [2]

Elements of resolution

We will not attach a complete resolution of this problem, because the reader will be interested in it

[1]in the sense that one, two or several « players » can play together, that possible actions are organized under rules (precise instructions), and that games are based on the use of some sort support, whether it is a material support, some data-processing or paper-pencil work. The « gaming environment » makes it possible to orient certain or all aspects of research situations in the direction where they can present problems in particular cases (under some choice of values of variables).

[2] G.Polya and Mr. Gardner in particular have studied this problem.

in order to get an idea of what it means to investigate such a problem. The pair (n, k) constitutes a variable of research. According to its value, the *progress* that can be made to solve the problem will be different. Indeed, the values of this variable can be classified in two categories, which correspond to different formulation and validation phases:

- case where there are several solutions: in this case, the formulation and the validation will consist in producing particular solutions, possibly supplemented by general methods of construction.

- case where there is no solution: solving the problem will consist in formulating the conjecture "there is no solution", and in validating this conjecture by means of mathematical arguments or by means of an exhaustive search of cases.

In addition, in order to progress in this problem, it is necessary to detach oneself from the actual colors and to consider the relative position of the sectors (or representing pawns), when compared to the others. That allows us to reach general methods of construction. One can introduce an additional variable, namely a shift, which can be defined differently according to the values of the pair (n, k). For example, in the case of the pair (n, n), the shift measures the position on the external disc with respect to the interior disc. One can thus obtain arguments of proof in the case where no solution exists.

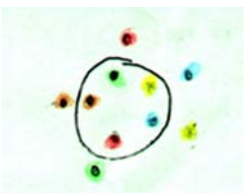
II.2. Conditions of experimentation

At the time we set up our experimentation, we made several choices. First, we chose to ask pupils to work *by groups of 3 or 4*. In addition, we provided the necessary *material equipment* to each group, in the form of two metal discs and magnets of different colors in sufficient number.

As the activity was planned in several sessions, we also gave to each group a *sheet of assessment* on which the pupils could record at any time the results of their research when they thought they were important, so that they could later rely on their written notes. Moreover, after several research sessions running under this setup, we organized a *joint session* with all groups, so that they could communicate and pool their results, their methods, their conjectures, and possibly hold further discussions. During this final meeting, the groups did not have the material equipment at disposal. There was only one available for the whole the class on which the colors were not chosen by the pupils, but which they could use to illustrate their matter.

We posed the same problem to students of 1st year of university, 11/12 and 9/10 years old pupils. The mathematical situation was managed by a researcher, while the teacher ensured the social management of the class.

II.3. Analysis of pupils' productions



Taking our observations into account, we consider that, except for two groups of 9/10 year pupils and one group of 11/12 year pupils which remained at the very first stage of the game, all the others entered a mathematically oriented, more or less elaborate, step of research.

At all levels, several values of the pair (n,k) were studied. Moreover, no outstanding differences appeared in the research dynamics of the different groups, although the school levels were different, except perhaps for an « inductive » method of research[3] which only appeared at the primary level. Several other methods of constructions of solutions have been proposed, whatever the level and studied cases were. They appear to be inspired by the research strategies. In the three secondary classes, the shifts were introduced by several groups. Even if they did not have organized steps of research, all the groups raised some conjectures. Some proceeded by grouping conjectures associated with particular cases ; others tended to raise more general conjectures. Among them, all managed to state the conjecture that there did not exist a solution whenever that was the case. However this appeared more easily at the university level. We make the assumption that this difference is possibly due to a more prevalent

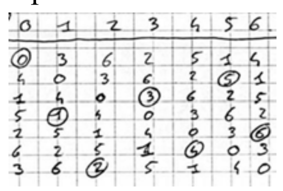
[3]It consists of keeping what works fixed and modifying the rest.

idea among pupils of secondary levels that a problem always has a solution, considering that this is true for the majority of the exercises which are proposed to them. At university, students can be confronted with exercises without solutions, e.g. in the case of the resolution of equations, and our students had in any case be faced to such issues through research situations they had studied beforehand. Finally, except for three university groups, all other groups relied on an exhaustivity approach (which was not a priori taken for garanted) in order to prove the absence of solution, and used very little proof arguments in those cases.

This situation thus worked quite well as a RSC within the directions that we defined. The training concerned corresponds to those situations which we previously evoked, such as to consider that a problem of mathematics does not inevitably have a unique solution, in fact it can have several ones or none at all. Other perspectives considered were to "decontextualize" the situation, working out methods of construction, seeking to generalize, but also stating conjectures, invalidating them by counterexamples, asking the question "why ?".

II.4. About material support and installation conditions

It appears that the existence of a material support facilitates the problem devolution. It also enabled the pupils to provide counterexamples whenever necessary, in order to give a basis to their conjectures. We thus make the hypothesis that the material support is a help with research because it permits a more direct handling.



However, even if the construction of methods of resolution that were discovered are similar, their formulation seems to have been influenced by the use of the material support. Among the university groups, those who used it at the beginning of their research quickly proceeded to paper-pencil work, introduced a numerical coding of the colors and a representation of the problem with a table. They gave methods of construction which tended to be detached from the concrete description, by using a mathematical vocabulary and by trying to generalize.

As we had supposed, the fact of working in group allowed students to debate, argue and avoid discouragement. Moreover, the game approach seems to enable pupils in difficulty to develop their argumentation, as they were led to discuss with other pupils which usually had better success in mathematics. They seems possible, ultimately, because this type of problems puts pupils at an "equal level of knowledge". The assessment sheets seem to help the pupils to judge what is important or not, and avoids them getting lost from one week to the other. Moreover, they show the importance of the clearly setting down what is going on, for the sake of re-using later the data. The joint session allowed the different groups to orally formulate their construction methods, which they had not necessarily succeeded to do in their written work. The fact that the choice of colors is not the pupils responsibility made it possible to invalidate research methods by gropings, to the profit of more organized research methods which were developed because they are more effective.

The presentation of one RSC using a suitable material support reinforces its accessibility at all levels, by facilitating the problem devolution. However, when the aim is to lead pupils to decontextualize and to generalize, it appears necessary to propose at least a session without any material support provided, or, at the very least, during a work sesssion for which it is not of any substantial help with research. We currently study how this "withdrawal" must be negotiated and what consequences can be attained from such attempts. In addition, we try to take into account the time variable and to establish under which conditions the repeated practice of RSC as a shared activity allows pupils to learn the various components of research activity in mathematics we underlined, and can influence their personal viewpoint with respect to mathematics.

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"RESEARCH" AS A WAY OF LEARNING NEW CONCEPTS: THE PARTICULAR CASE OF CONSTRUCTING DEFINITIONS

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We propose to consider new situations for the study of the formation of mathematical concepts. This paper is based upon mathematical research, and on the process of definition-construction in particular. We shall explain our choices and give an overall survey of the situations of definition-construction (called SDC) in which we have conducted experiments.

Foreword: the experiments described in this paper have been carried out with first year university students, but they are relevant to the high school students as well.

"DOING MATHEMATICS"

When the construction of a new concept is being discussed, professional mathematicians could point out the specificity of defining as a mathematical activity. However, although definition-construction has an obvious place in mathematical research, it seems that next to nothing has been written on the subject of "definition-construction" in the relevant literature, neither in pure mathematics, nor in mathematics education or in the different curricula. Indeed we could make the following remarks: firstly, Lakatos (1961) developed in parallel the formation of a concept and the construction of its definitions (using the method of proof and refutations). Secondly, the situations of classification (as proposed and analysed by Freudenthal (1973) as defining activities and by Mariotti & Fischbein (1997)) and redefining situations (as proposed by Borasi (1992), when redefining "circle" in Taxicab Geometry for instance) can be studied from the definition-construction point of view. But the classification and redefining tasks are only the tip of the iceberg consisting of "definition-construction situations". We will propose different types of situations of definition-construction (called SDC) from now on. Thirdly, we notice a specific point in French curricula, included in the expression "doing mathematics". Although there is no consensus about what "doing mathematics" means for mathematicians (it depends in fact on each individual), we can still underline the main features of our idea of "doing maths". Of course, it includes specific and transversal knowledge and skills[i], such as the following: proving, conjecturing, refuting, creating, modelling, extending but also transforming a questioning process, being capable of non-linear reasoning, having a scientific responsibility and above all defining. All these points may have a place in French curricula under the (unique) key word "scientific activity".

We assume that definition-construction situations represent an "opening" for the construction of concepts at school, which can be integrated in curricula. We have now to substantiate this fact, so, the aims of this paper are: to define SDC (Situations of Definition-Construction), to characterise mathematical objects propitious to SDC, to grasp "unanswerable" elements to bear out the potentiality of SDC for the formation of concepts.

We use the expression "definition-construction" naturally, but what is its exact mathematical meaning? What are the theoretical elements, required for the mathematical explanation and the design of such situations in the classroom?

CHARACTERISATION OF SDC & THEORETICAL BACKGROUND

At the beginning of our research, we started from a "simple" definition of SDC i.e. "situations whose resolution involved the construction of definitions". It means, in particular, that there is no predetermined solution to a certain type of problems, but an optimal resolution with definition-construction. The study on SDC seems to present a great complexity due both to the difficult concept of "definition" (from a mathematical point of view) and to the lack of SDC in curricula and textbook (from an educational point of view). We assume that a specific theoretical background is required for the study of such novel situations, in order to build and analyse both the definition-construction and the concept-formation. It doesn't imply necessarily that theoretical tools, used by Freudenthal, Mariotti-Fischbein or Vinner for instance, should be dismissed, but the core of our research on SDC consists in proposing a theoretical background,

- which is appropriated to every type of SDC described below and to every field of mathematics (i.e. its use shouldn't be limited to classification or redefining tasks and to a certain mathematical field such as geometry),
- which can describe a defining process both from an mathematical and a didactical point of view,
- which can include students' pre-existing conceptions on definition (we assume that SDC should bring about an evolution of conceptions on the concept of definition).

Let us now specify key elements of characterisation of SDC and the theoretical tools we chose. An epistemological and mathematical study of the concept of "definition" leads us to a typology of SDC; we can identify three main types, which can be described as follows:

- CLASSIFICATION consists in delimiting what characterises a concept, starting from examples and counter-examples for instance.
- MATHEMATISATION/MODELLING: the characterisation of this type of SDC is initiated by its name. Let us propose an example: "define a mathematical object, which can represent the set of plants" (i.e. the elements of the whole vegetable kingdom). It could correspond to the mathematical concept of "tree".
- PROBLEM-SITUATION: this expression was used by Lakatos (1961) and was not exactly defined, but according to Lakatos, starting from a vague idea of a mathematical concept (such as Euler's formula for the study of polyhedra) can be enough for marking the beginning of a definitional procedure (Lakatos,1961, p.69). It comes close to our first natural definition.

The previous typology was established having regard to an identification of three emblematic conceptions: the Lakatosian, the Aristotelian and the Popperian conceptions. The choice of these conceptions was made in accordance with three main "levels" of the defining process i.e. dialectic between definitional procedure and concept-formation (Lakatos & Problem-Situation), language and logic (Aristotle & taxonomy tasks), and axiomatic (Popper & construction of new theories).

The characterisations of both typology and conceptions were written in order to plan, describe, grasp and explain the construction of definitions. That's why we used the cK¢ model[ii] (Balacheff): this model allows an identification of stages in the construction of definitions; it brings elements, which can help us grasp the formation of concepts, and it can include students' pre-existing conceptions on definition. We develop this theoretical background and the theorisation of the Lakatosian and Aristotelian conceptions in Ouvrier-Buffet (2002b&2004) (*).

Lakatos' work offers an interesting model of production of new knowledge, and his thesis discusses the complex dialectic between a definitional procedure (so definition-construction and concept-formation) and proof. In order to make a long story short, let us give a foretaste of this research. We remind here some key words of the promising Lakatosian view and their characterisation: *zero-definition*, *proof-generated definition*, *refutation*. We will present an overall characterisation of the Lakatosian conception with the cK¢ model. A conception is characterised, in the cK¢ model, by: a set of problems (here Problem-Situation), a set of *operators* and a set of *control-structures* (the last two undefined terms can be used in their common sense for a start), and a system of representation (in this case, the system of representation involved a mathematical research activity).
A *zero-definition* is a tentative definition emerging at the beginning of the investigation. It may evolve into a *proof-generated definition* or just disappear. It is brought about by proof and stands out as the most important notion in Lakatos' view: the product of *proof-generated definition* is directly linked to the type of SDC (i.e. Problem-Situation, according to Lakatos). Of course, the method of refutation is central to Lakatos' development. In particular, from the definition-construction viewpoint, several *operators* can be mobilized: they could consist in generating counter-examples, coming back to the generation of a proof, and, to a minor extent, making linguistic, logic and axiomatic demands: this last *operator* is specifically representative of both the Aristotelian and the Popperian conceptions. The Lakatosian set of *operators* contributes to the

evolution of a *zero-definition*. Hence, the notion of *zero-definition* is promising and useful for the characterisation of the start of a definitional procedure and its evolution. How does Lakatos *control* his definitional procedure? To answer this question presents a real complexity, in particular because both notions of *zero-definition* and *proof-generated definition* are defined by Lakatos each with regard to the other. Thus one *control-structure* of the definitional procedure mobilizes the proof. This aspect needs to be studied.

We complete this current characterisation of SDC by specificities of mathematical concepts involved in our experimented situations: we design three SDC with objects coming from discrete mathematics. Their main features are that these objects are accessible by their representations, they are not institutionalised[iii] (so the students have no a priori expectations when they work on these objects), and these objects can be worked on in different ways.

OUR SDC

We have experimented two main types of SDC. The first one was CLASSIFICATION: two situations were designed as described below. One of them involves the mathematical object "tree" (see Ouvrier-Buffet 2002a) and the other the object "discrete straight line". The main results concerning students' construction of definitions are mentioned below.
The second type of SDC was PROBLEM-SITUATION: the mathematical concepts at stake in such situations can be variable. A first experimentation was based on "displacements on a regular grid map" (see Ouvrier-Buffet 2002b for the description of this situation): it allows a natural problematisation of concepts of "generator" and "minimality" (it means a problematisation in Z of concepts generally taught in vector spaces). The second experimentation involves "discrete triangles": we propose to describe the main features and results of this experimentation compared with results from the other type of SDC CLASSIFICATION observed when watching the students.

SPECIFICATION OF STUDENTS' DEFINITION-CONSTRUCTION PROCEDURE

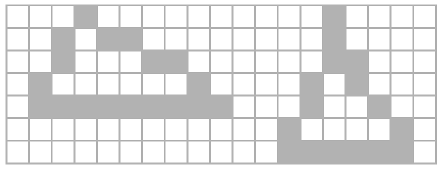
Classification with an explicit request of definition (tree and discrete straight line)

The statement of such a situation contains a question ("how could you define…?") and examples and counter-examples (identified as such or not) of a mathematical object.
The resolution by students of the two experimented SDC (type CLASSIFICATION) is made up of the following main elements: the major defining procedure used by students is based on the Aristotelian method of definition by *genus and differencia*. Moreover, students use given examples and counter-examples in order both to determine characteristics of "tree" or "discrete straight line" and to test their current definition. These comparison and test mechanisms seem natural because students' task consists in defining, delimiting a mathematical object starting from examples and counter-examples. Hence, a reflexivity on *zero-definitions* comes from refutations with counter-examples on one hand, and from linguistic and logical considerations on the other. We underline the fact that students are able to deal with different approaches to the mathematical object: and, they question the implication and the equivalences between the different points of view tackled.

Problem-situation (discrete triangle)

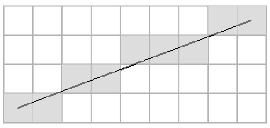
The SDC was proposed to two groups of students (first university year, scientific section) as follows: "one wants to colour squares on a regular grid map. Draw triangles (colouring squares). Explain your construction."
No explicit request of definition(s) was formulated. But such a situation carries the need of definition-construction (discrete triangles), with primitive elements (discrete straight line or segment). There are more ways than one to solve this problem-situation. It is possible to use different approaches to the mathematical discrete object and thus different ways to build it appear, e.g. starting from real straight line and one of its definitions or constructing such a discrete triangle starting from three given pixels (*).

Note that the perceptive aspect of discrete drawn objects is important and non-neutral in the students' debates we have observed. This picture shows how the perceptive round or hollow aspect of a triangle can play a part in a work on the definition and the construction of such discrete objects.



The problematic of construction is coupled with an axiomatic problematic i.e. to define discrete straight line requires questioning the existence (and uniqueness even) of the intersection of two discrete straight lines: this last point adds to the difficulty of the task a priori (*). We focus now on the geometrical object "discrete straight line".

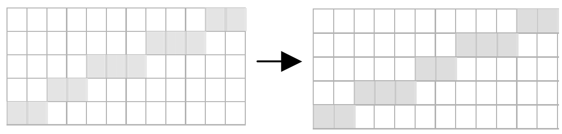
Different *zero-definitions* of discrete straight line are conceivable: a first group of *zero-definitions* is connected to an approach of this discrete object by the real object. We have three *zero-definitions*:



Zdef1: set of the pixels crossed by a real line;
Zdef2: set of the pixels "**the nearest"** of a real line;
Zdef3: set of the pixels "**inside"** a band.

Should you not draw another discrete straight line, when the above real straight line is given? This question underlines the potentialities for evolution of a *zero-definition* (among the above-mentioned). This mathematical analysis of potential evolution of conceivable *zero-definitions* is crucial when designing SDC. It determines and marks out ways for concept-formation.
Two other *zero-definitions* can be enounced, starting from a "regularity" view-point i.e. considering that the repartition of the pixels of a discrete straight line should be regular (a word to be defined! Again, this attests the potential evolution of the following *zero-definitions*):

Zdef4: sequence of stages of pixels with specific **properties**;
Zdef5: sequence of pixels stages with a uniform repartition, **non-improved** from the regularity viewpoint.



Remarks on observation of students' procedure (*)
Students' work on the construction of discrete triangles was very dependent on the perceptive aspect of discrete objects. Beyond this stage, different potential *zero-definitions* of "discrete straight line" were produced by students: they both use the "real straight line" and the "regularity" approaches. Students switch from one approach to another one. Eventually, only "regularity" problematic was retained, because students actually try to fit in within the discrete framework without any outside reference. That's very interesting when we consider the scientific approach. Nevertheless, students didn't work on equivalence between their potential *zero-definitions* and didn't identify the task as a "definitional task". Thus, all the *operators* in relation with linguistic and logical aspects were not mobilised and we notice that the evolution of *zero-definition* was blocked. In fact, students don't assume the responsibility of the construction of definition and they don't mention it. Nor do they engage in a test and a refutational process using counter-examples such as students working, or having worked, on classification tasks would do.

POTENTIALITIES OF SDC

The explicit request of definition in classification situations allows a reflexivity on the students' construction of definitions, in particular, the reflexivity on their current "definition" which causes this definition to evolve. SDC Classification constitute a good beginning for "definition-construction". In other respects, a Problem-Situation requires more time: it is a more difficult type of SDC and it seems necessary that a SDC type CLASSIFICATION precedes a PROBLEM-SITUATION in order to prepare students to a "first" contact with definition-construction search.
The main positive points of SDC are as follows:

- SDC are an opportunity of a work on scientific process (construction of definition and more: proof for instance). This scientific process is constituted by the students' exploration of different approaches, doubting, conjecturing, refuting (generating new counter-examples), testing etc.

- A work with SDC allows an enrichment (and generates modifications) of students' conception on the concept of definition in mathematics.
- SDC are a place for the formation of mathematical concepts: this fact is attested often by *zero-definitions* evolution, and not often by *proof-generated definition*.

Nevertheless, we can point to some limits to SDC. In particular, in the Problem-Situation, we have observed no evolution of potential *zero-definitions*, and the presence of definition-in-action were sufficient for a part of the resolution of the problem. We would like to point out that the explicit request for definition is pertinent in classification situations because it allows a reflexivity on definition-construction work through language and logic. But one of the main aims is for students to become "apprentice-mathematicians", to be able to mobilize scientific activity, so that they develop abilities and autonomy to construct definitions and mathematical concepts. Eventually, we have to suppress the explicit request for definition: hence, we have to design a set of SDC with a progression from the explicit request for definition in a classification situation to an open Problem-Situation without any indications about a "definitional task".

FURTHER OUTLOOK

At the moment, we have three main perspectives for further research. The first one consists in studying thoroughly proof-generated concept and *proof-generated definition* in order to build situation in which definition-construction is required (at the same time, such situations mobilize definition-construction and a reinvestment of produced definitions, in a dialectic interaction with a proof). The existence of this dialectical process is apparent in Lakatos' work but not described.

The second perspective concerns the extension of SDC to institutionalised concepts (and not only redefining situations, in a different context, with known concepts). The objects from the geometrical field seem to be worthwhile because they share common characteristics with discrete objects. We have some ideas to experiment (example: up to what extent can a convex (object) be "naughty"? Or how can we characterise the 'least round' convex shape?).

In order to integrate SDC in the curriculum as situations, which make a work on concept-formation possible, we have to precise how to conduct such situations. We don't specify in this paper the place and the role of the teacher, but we have elements for the characterisation of the authorized interventions of the teacher: neutrality and follow-up interventions (asking for a written definition, asking for new examples in order to focus students' research on a characterization, asking for counter-examples, and proposing counter-examples if (and only if) the situation is "blocked").

(*) ADDITIONAL INFORMATION IS AVAILABLE ON REQUEST

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[i] i.e. skills and knowledge which straddle mathematics, used in the whole variety of mathematical contexts.
[ii] "cK¢" are the initials of "**c**onception", "**k**nowledge" and "**¢**oncept".
[iii] An institutionalised concept is a "curriculum" concept i.e. a concept that has a place in the classic taught content.

A PHENOMENAL APPROACH TO MATHEMATICS

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INTRODUCTION

Mathematics is often presented as a collection of disparate topics, each with its associated techniques, typical problems, concepts and language, images and classic student errors (Griffin & Gates ref), Mason & Johnston-Wilder 2004). Each topic is the distillation of a method which someone found for solving a class of problems. As with arithmetic, where the four operations are abstracted from contexts and only later are contexts reintroduced, methods constituting topics are often isolated and abstracted from any originating context. Only towards the end of the study of the topic are 'applications' introduced. Since learners are tested on the use of techniques on routine questions, teachers are tempted to follow textbooks in demonstrating techniques using worked examples, and then inviting learners to follow the template, or as Gillings (1961) reports from Egyptian papyry of 200 BCE, to 'do thou likewise'.

In order to provoke learners into taking initiative, into engaging fully with mathematical ideas and mathematical thinking, it is necessary to construct pedagogic tasks which call upon learners to make use of their undoubted powers of making sense. Those powers include

imagining & expressing what is imagined;

particularising, specialising, & generalising (Polya 1962, Mason *et al* 1984);

conjecturing & convincing yourself and others;

organising & characterising;

focusing and de-focusing.

These powers are involved in all of human sense making, but form the core of mathematical sense making. For descriptions and examples of these and similar powers, see Dewey (1938) Cuoco *et al* (1996), Mason & Johnston-Wilder (2004).

Many authors have observed that what activates human sense making is disturbance, experienced as surprise, as puzzlement or perplexity (Dewey *op cit*, Heidegger 1962), as recognition that conscious or unconscious expectation has been broken, or as dissonance, whether cognitive (Festinger 1957), affective, or enactive. Paul Halmos makes use of this by always beginning with a question:

> Let me emphasize one thing ... the way to begin all teaching is with a question. I try to remember that precept every time I begin to teach a course, and I try even to remember it every time I stand up to give a lecture... [Halmos 1994 p852]

The claim put forward here is that mathematical thinking can be initiated through experiencing some phenomena which trigger questions. These phenomena may be drawn from the material, imaginal, symbolic, and social worlds which human beings inhabit (corresponding to Bruner's three modes of representation: enactive, iconic and symbolic with the addition of the social). Once initiated, that thinking needs to be fostered and sustained through the use of mathematically conducive ways of working, into which learners are socially enculturated through being in the presence of a teacher who not only manifests relevant behaviour, but is aware of that behaviour and uses techniques to promote that behaviour in learners. Ways of working has been variously described: *How To Solve It* (Polya 1957, 1965), *scientific debate* (Legrand 1993 see also Mason 2001), a *conjecturing atmosphere* (Mason *et al* 1982), *ways of working* (Tahta & Brookes 1966) and so on. And as Polya pointed out, most vital for actual learning as distinct from mere participation is the phase of *looking back*, of making sense of what has been done, of entering as fully and vividly as possible key moments of the work (Tripp 1993) in order to strengthen links between the state of being stuck and strategies for getting unstuck (Mason *et al op cit*, Mason 2002).

In this short paper I will only be able to offer a number of examples of phenomena which invoke learners' mathematical sense making and which illustrate some of the possibilities. In my presentation I shall use different examples. The underlying conjecture is that associated with every topic there is a surprise or disturbance, as well as a frisson of pleasure at being able to encompass a whole class of problems in a single method or way of thinking. The teacher's job at the beginning of a topic is to re-experience, to re-enter that surprise in and for themselves (Moshovits-Hadar 1988), in order to create conditions in which learners too can experience that initiating energy.

SOME SAMPLE PHENOMENA

ROLLING CUP

Phenomenon A plastic cup rolls about on the floor.

Questions What path will the cup follow, and what dimensions of the cup do you need to know in order to predict details of the path? Could two cups of different shapes roll on the same path, in some sense?

Possible developments Predicting the path shape is based on experience of the world, though it is not so easy to be precise about why it must have that shape. Predicting details of the curve makes use of properties of circles and the use of ratios. There are opportunities to work on seeking and expressing relationships and on generalising rather than simply dealing with a particular cup.

Elaboration

A simple phenomenon, observed by many people, many times, nevertheless has the potential for offering learners experience of mathematising. The real issue with techniques is not so much to learn to apply them, which is a pedagogic issue, as to recognise that a particular technique could be of use: knowing-to use a technique is psychologically quite different from knowing-how to use it, or even, given the technique, knowing-when to use it. Knowing-to requires something in a situation to bring a relevant technique to mind.

Drawing a diagram is part of the movement from material or experienced phenomenon to the imagistic world of diagrams and icons, on the way, usually, to the symbolic world of formulae and algebraic or other expressions. In order to draw a relevant diagram certain features have to be stressed (e.g. the cross-section where the cup touches the floor) and others consequently ignored (the colour and material of which it is made). Furthermore the circularity of the rim and bottom are integrated into the awareness that what matters is the ratio of the top and bottom rims and the vertical height between them. The diagram then needs to be extended to show where the centre of the rotation will be as the cup rolls about, permitting some symbolic calculations in order to determine the radius of the rotation circle (whether measured at the top or bottom rims).

MOVING FINGER

Phenomenon A metre rule or other similar stick rests on one finger on each hand. The two fingers move towards each other smoothly and uniformly. Watch the stick's movement.

Questions Why does that happen?

Possible developments Clearly friction is relevant, and centre of gravity is likely to come into any explanation. There are opportunities to imagine what will happen before it starts, then to try reconcile prediction with observation: the role of mathematical modelling. Is there a two dimensional version?

SAUSAGES & CHOCOLATE BARS

Phenomenon A number of identical sausages are to be divided fairly among a number of people.

A number of chocolate bars are to be divided fairly among a number of people.

Questions What is the least number of cuts?

Possible developments Modelling assumptions that ends of sausages are no different from middle bits, perhaps; Opportunity to compare methods, to widen appreciation of ways of depicting fractions; contact with extremal problems.

MIDPOINT

Phenomenon Imagine a parabola, and chord between two points on the parabola. Pay attention to the midpoint of the chord.

Questions What are possible positions of the midpoint as both ends of the chord move freely on the parabola.

Possible developments What about for a cubic? Suddenly all sorts of preconceptions about curves surface. Related questions include assigning to each point of the plane the number of tangents to a given cubic through that point, and looking for the boundaries of regions of points with the same value. This can be extended to quadratics and indeed to any differentiable function.

PARABOLA

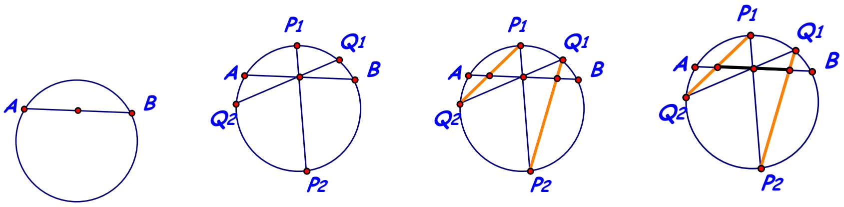
Phenomenon Imagine four points on a parabola. Draw the straight line through each pair of points. The lines meet in pairs in seven points, four of which are the original ones. Draw the circle through the remaining three points and note the position of the centre. Drag or otherwise move the four points about on the parabola.

Questions What is the locus of the centre of the circle?

Possible developments Use of coordinate geometry. What is special about the parabola?

CIRCLES

Phenomenon



Questions Is the point which is the midpoint of AB, of P1Q1, and P2Q2 always the midpoint of the segment cut off by P1Q2 and P2Q1 on AB?

Possible developments Looking for relationships, reasoning from definitions

Looking for generalisations and-or extensions

NUMBER PATTERNS

Phenomenon $1 = 1^3$, $3 + 5 = 2^3$, $7 + 9 + 11 = 3^3$, $13 + 15 + 17 + 19 = 4^3$, …

Questions Does this continue? What is the 'this' that continues? Is it always true?

Possible developments Are there some variations. Opportunities to imagine, to detect relationships and express these as general statements; opportunity to use symbols to express relationships.

(-1) X (-1) = 1

Phenomenon From a spreadsheet or other animation, some entries appear in a grid as shown below

				1 x 3 = 3		
			0 x 2 = 0	1 x 2 = 2		
			0 x 1 = 0	1 x 1 = 1	2 x 1 = 2	3 x 1 = 3
			0 x 0 = 0	1 x 0 = 0		

Questions What entry would you expect in cells adjacent to those shown?

Extend the cell entries upwards and to the right until you can predict the entry in any cell above and to the right of 0 x 0 = 0.

Extend those cell entries downwards. Extend those cell entries to the left.

Now extend the cell entries above and to the left of 0 x 0 = 0 downward; extend the cell entries below and to the right of 0 x 0 = 0 to the left: is there consistency when you extend to the same cell from two directions?

Possible developments Other grid patterns can be established.

IMAGINE

Phenomenon Imagine a point on the plane. Imagine a circle which is a fixed distance from that point. Now imagine two points, and a circle which is equidistant from both points. Now imagine three points, and a circle equidistant from all three points.

Questions In each case, what about the circle can change and still preserve the condition? Where can the centre of the circle get to under the specified condition?

Possible developments Four points? Variations produce classic geometric theorems such as the incircle and the circumcircle of a triangle.

MEAN-MEDIAN-MODE

Phenomenon On a spreadsheet is displayed a histogram of some data points, the mean, median and mode, and possibly the standard deviation. Changing the data points changes the display immediately.

Questions How far apart can the mean, median and mode be within a specified range? How many points can be outside I standard deviation of the mean?

Possible developments Constructing 'data' with specified combinations of mean, median, mode and standard deviation.

FACTORING QUADRATICS

Phenomenon

$x^2 + 5x + 6 = (x + 2)(x + 3)$	$x^2 + 5x - 6 = (x + 6)(x - 1)$
$x^2 - 5x + 6 = (x - 2)(x - 3)$	$x^2 - 5x - 6 = (x - 6)(x + 1)$
$x^2 + 10x + 24 = (x + 6)(x + 4)$	$x^2 + 10x - 24 = (x + 12)(x - 2)$
$x^2 - 10x + 24 = (x - 6)(x - 4)$	$x^2 - 10x - 24 = (x\ -1\ 2)(x + 2)$
$x^2 + 17x + 60 = (x + 12)(x + 5)$	$x^2 + 17x - 60 = (x + 20)(x - 3)$
$x^2 - 17x + 60 = (x - 12)(x - 5)$	$x^2 - 17x - 60 = (x - 20)(x + 3)$

Questions Could more be generated?

Possible developments Characterise all quadratics for which all four possibilities of signs of the coefficients factor. What about quadratics with lead coefficient greater than 1? (Minor 1988, Kaczkowski 2001)

There are opportunities to imagine beyond what is seen, to extend and to express generality make this a powerful phenomenon.

Showing some objects, sometimes dynamically, sometimes statically, and asking learners for what is the same and what different, what is changing and what is invariant, is likely to intrigue them sufficiently to want to explain what things happen as they do, or how some pattern might continue. Some phenomena are observed in the material, imagined (including e-screens), symbolic, or social world. Others are created by recognising and exploiting some aspects of the surprise which occasioned the topic in the first place.

PHENOMENAL ONTOLOGY AND AFFECT

What makes me ask questions like these? It is certainly a propensity of mine, but all I do is notice when I am surprised by something, and then ask about the source of that surprise. This is something that everyone can develop. As you pay attention to what surprises you, you also find that you are surprised more and more.

What do I mean by *a phenomenon*? That it is difficult to capture in words is evident from Husserl:

> … the infinity of actual and possible world-experience transforms itself into the infinity of actual and possible 'transcendental experiences', in which, as a first step, the world and the natural experience of it are experienced as 'phenomenon' [Husserl 1970 p153]

I mean some incident or event which is salient and identifiable, and hence which is discerned from the background of experience. By the time something is recognised *as* a phenomenon, there is awareness of generality, of actual or potential similarity and replicability in some form. A phenomenon is created by the observer who discerns, who foregrounds and backgrounds, who stresses and ignores. Other people may not discern in the same way, and even if they discern, they may not be aware of having noticed.

Phenomena, like 'problems' are not intrinsically interesting. It is people in a situation who are interested. Once something discerned is recognised *as* a phenomenon, interest has already been aroused.

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FURTHER TEXT

Material world; mental world, symbolic world; virtual world

Static, dynamic

Must-May-Can't happen

TREATING EXAMPLES

HALF MOON INN

Phenomenon Some years ago I was driving to a workshop I was to give, when I noticed a pub sign similar to the one shown here. The pub was called the half moon inn. Something about the sign caught my attention: something was not right.

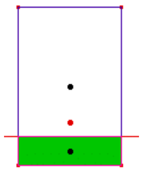
Questions Can you ever see a vertical half moon, in the sense that the terminator is a straight line in line with your body as you stand up straight and look at it? If so, when and where would you look; if not, how close can you get?

Possible developments Related problem: can you ever see a horizontal half moon. If so, from where, if not, how close can you get?)

Opportunity to work on imagining and expressing, conjecturing and convincing, as well as to make observations.

TIPSY

Phenomenon One Before you start drinking from a can, the centre of gravity of the can and the liquid is in the middle; as you drink, the centre of gravity drops, following the drop in the surface of the liquid; when you have drunk all the liquid, the centre of gravity of the can is back in the middle where it began.



Questions Somewhere the centre of gravity reaches a minimum height. Where is it?

Possible developments Use of reasoning (no measurements need be made; opportunity to use calculus to find extremal value; use of modelling assumptions.

Phenomenon Two I was playing with a coke can while listening to people talking over lunch, and I accidentally discovered that with only some of the liquid in it, the can balances on its rim. The presence of the liquid gives it an odd motion as it rolls around at this angle.

Questions Somewhere the centre of gravity reaches a minimum height. Where is it?

Possible developments I immediately wondered over what range of volumes of liquid the can would remain stable in this position. What is significant for our purposes here is again, not a model devised to answer this, but rather the sorts of choices made.

Phenomenon Three

Work on the drinks can put me in mind of wine racks that I have often seen in shops but never really paused to think about before.

Questions

What are the design constraints that make this thing stable for both full and empty bottles?

Possible developments

Centres of gravity; combining centres of gravity of different objects to find the centre of gravity of a compound object.

CHECKOUT

In a supermarket, when is it most efficient to start up another checkout?

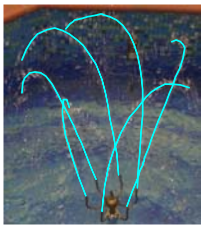
FOUNTAINS

Phenomenon One

Scotch College is an independent school in Melbourne, where I encountered an atrium in the faculty building with a fountain in the centre. The fountain consists of six spouts symmetrically placed at the centre as a rotating wheel. The wheel of spouts rotates, whether solely due to the water, or due to some other mechanism I do not know.

Questions

I looked at this a few times, and found myself asking: what am I really seeing? My brain expects parabolae, so that is what I saw … at first. In the second picture I have joined up drops which seem part of the same 'curve'.



Possible developments

Start with an ordinary static fountain. It would be possible to use pictures, even video, with the path of single droplets traced, leading to physics modelling of trajectories, then use of calculus or completed square for finding the apex, and the many related problems.

One aim would be to alert learners to the shapes of fountains so that they begin to notice different fountains for themselves, and to be aware of the ubiquitous presence of parabolae.

TIED DOWN

Phenomenon One

Consider tying a rope around a rod in a clove hitch. It is a very simple and common knot.

Questions

By how much will the rope be shortened?

Possible developments

Contrast between empirical approach through collecting data and fitting a curve through it, and a theoretical approach in which an underlying mathematical model is used to make predictions. Newton displayed both of these, the former in his Optics, and the latter in his Principia (Buchdahl 1961).

FUNNEL

Phenomenon	A circle can be folded in half, twice, and then opened to make a conical funnel.
Questions	How much does the funnel hold? How big a cone can be made from a single sheet of A4 paper?
Possible developments	Comparing volumes of cylinders made from a piece of A4 paper. What shape of curve must I cut along to get an elliptical cone from a sheet of paper?

Finding phenomena which can serve as entry to mathematical topics is one source of examples; starting from a topic and seeking relevant phenomena is another. This is most usefully done by looking for the fundamental surprises which underpin the topic: what is it that is not obvious, making this worthy to be considered a topic?